The Mean Value Theorem

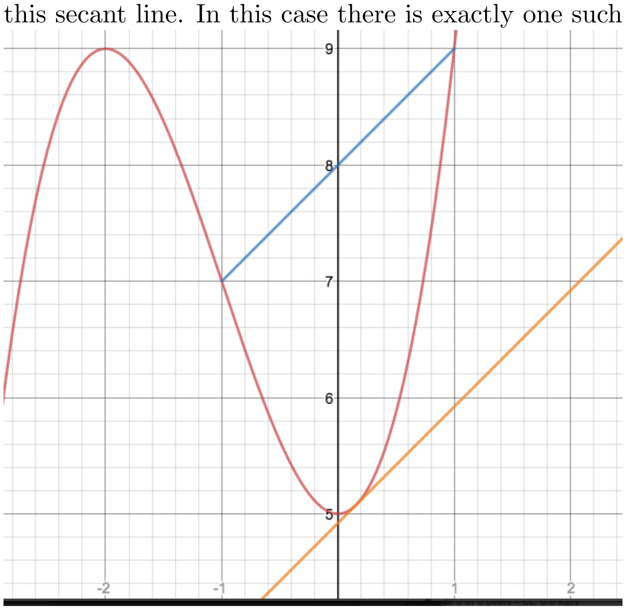
Let $f(x)$ be a function which is continuous on a closed interval $[a, b]$, and differentiable on the interval (a, b). Then there is at least one number c in the interval (a, b) such that the instantaneous rate of change $f'(c)$ is equal to the average rate of change of f over the interval $[a, b]$:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Illustration:
Consider the function $f(x) = x^3 + 3x^2 + 5$ on the interval $-1 \leq x \leq 1$. The function is continuous and differentiable on that interval (in fact, everywhere) so the Mean Value Theorem applies: there is at least one number c which is between -1 and 1, and $f'(c) = \frac{f(1)-f(-1)}{1-(-1)} = \frac{9-7}{1+1} = 1$

In fact, we can find the value of c:
$f'(x) = 3x^2 + 6x$
$f'(c) = 1 \implies 3c^2 + 6c = 1$
Solving $3c^2 + 6c = 1$ by the quadratic formula gives $c = -1 \pm \frac{2\sqrt{3}}{3}$. Only one of these lies in the interval $[-1, 1]$, namely, $c = -1 \pm \frac{2\sqrt{3}}{3} \approx 0.155$

If you look at the graph below which I made in Desmos, the function f is given by the red curve. The secant line connecting the points $(-1, 7)$ and $(1, 9)$ is drawn in blue. The MVT guarantees that there is at least one point on the graph of f in between those two points where the tangent line has the same slope as this secant line. In this case there is exactly one such point, and I have drawn its tangent line in orange.



Rolle'sTheorem

Rolle's Theorem is a special case of the Mean Value Theorem.

> Let $f(x)$ be a function which is continuous on a closed interval $[a, b]$, and differentiable on the interval (a, b), and which takes the same value at both endpoints: $f(a) = f(b)$. Then there is at least one number c in the interval (a, b) such that $f'(c) = 0$.

Illustration:
Consider the function $f(x) = x^2 - 3x - 10$ on the interval $-3 \leq x \leq 6$. The function is continuous and differentiable on that interval (in fact, everywhere) and $f(-3) = f(6) = 8$, so the Rolle's Theorem applies: there is at least one number c which is between -3 and 6, where the derivative is 0.

In fact, we can find the value of c:
$f'(x) = 2x - 3$
$f'(c) = 0 \implies 2c - 3 = 0 \implies c = \frac{3}{2}$, which is between -3 and 6, as promised.

A way to think about Rolle's Theorem is this: we have a function which has the same value at two points, and it is continuous and differentiable in between those two points. So (if the function is not constant) its graph must turn around and come back up or back down somewhere in between those two points, at least once. Since the function is differentiable, that means that there is at least one place where the derivative is 0 (a local maximum or minimum).