Find the radius and interval of convergence for each of the power series. Show all your work: simplify your result, reduce fractions to lowest terms if necessary. No decimals. You must show at least some work in each problem in order to receive credit. Put a circle around your final answers.

1) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}n}{4^n}(x-3)^n$

$$\begin{aligned} \left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{(n+1)(x-3)^{n+1}}{4^{n+1}} \cdot \frac{4^n}{n(x-3)^n}\right| \\ &= \frac{n+1}{4n}|x-3| \end{aligned}$$

$\displaystyle\lim_{n\to\infty}\left(\frac{n+1}{4n}|x-3|\right) = |x+3|\lim_{n\to\infty}\left(\frac{n+1}{4n}\right) = \frac{1}{4}|x-3|$

The series will converge if $\frac{1}{4}|x-3| < 1 \implies |x-3| < 4$, so the radius of convergence is 4.

$|x-3| < 4 \implies -4 < x-3 < 4 \implies -1 < x < 7$

We investigate the endpoints:

When $x = -1$, the series is $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}n}{4^n}(-4)^n = \sum_{n=1}^{\infty}(-1)^{n+1}n(-1)^n = \sum_{n=1}^{\infty}(-1)^{2n+1}n$

Note you could simplify this, because $2n+1$ is always an odd number, so $(-1)^{2n+1} = -1$ for all n. So $\sum_{n=1}^{\infty}(-1)^{2n+1}n = \sum_{n=1}^{\infty}(-n)$ This diverges, because the terms do not go to 0: $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty}(-n) = -\infty$. So $x = -1$ is not in the interval of convergence.

When $x = 1$, the series is $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}n}{4^n}(4)^n = \sum_{n=1}^{\infty}(-1)^{n+1}n$

This diverges, because the terms do not go to 0:
$\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty}(-1)^{n+1}n$ does not exist: the terms of the sequence oscillate between values which are going to $-\infty$ and values which are going to $+\infty$. So $x = 1$ is not in the interval of convergence.

So the interval of convergence is $-1 < x < 7$, or in interval notation $(-1, 7)$

2) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{(n+1)2^n} x^n$

$$\begin{aligned} \left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{x^{n+1}}{(n+2)2^{n+1}} \cdot \frac{(n+1)2^n}{x^n}\right| \\ &= \frac{n+2}{2(n+1)}|x| \end{aligned}$$

$$\lim_{n\to\infty}\left(\frac{n+2}{2(n+1)}|x|\right) = |x| \lim_{n\to\infty}\left(\frac{n+2}{2n+2}\right) = \frac{1}{2}|x|$$

The series will converge if $\frac{1}{2}|x| < 1 \implies |x| < 2$, so (the radius of convergence is 2.)

$|x| < 2 \implies -2 < x < 2$
We investigate the endpoints:

When $x = -2$, the series is $\displaystyle\sum_{n=0}^{\infty} \frac{1}{(n+1)2^n}(-2)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)}$
This converges, by the Alternating Series Test: $|a_n| = \frac{1}{n+1}$ is decreasing and its limit as n goes to infinity is 0. So $x = -2$ is in the interval of convergence.

When $x = 2$, the series is $\displaystyle\sum_{n=0}^{\infty} \frac{1}{(n+1)2^n}(2)^n = \sum_{n=0}^{\infty} \frac{1}{(n+1)}$
This diverges: there are a number of ways you could show this - pick one and carry out the details! You could use Limit Comparison with the harmonic series. (Notice that you cannot use direct comparison with the harmonic series, because $\frac{1}{n+1} < \frac{1}{n}$). Another way is to use the Integral Comparison Test. Or, you could observe that this series is just the harmonic series with the first term cut off. By any one of these three means, the series diverges. So $x = -2$ is not in the interval of convergence.

So (the interval of convergence is $-2 \leq x < 2$, or in interval notation $[-2, 2)$)