The extreme value theorem:
If a nonconstant function $f(x)$ is continuous on a closed interval $[a, b]$, then $f(x)$ has an absolute maximum and an absolute minimum on that interval.

Notes:

- The absolute maximum and absolute minimum are only for values of $f(x)$ in the interval $[a, b]$. It is possible that $f(x)$ has higher or lower values outside that interval. For example, $f(x) = 3x^4 - 16x^3 + 18x^2$ has an absolute maximum on the interval $[-1, 4]$ but it does not have an absolute maximum on the whole real line.
- Find examples to help you understand why each of these is necessary in the theorem:
  - $f(x)$ has to be nonconstant,
  - the interval must be closed, and
  - $f(x)$ must be continuous on the interval.

Fermat's Theorem:
If $f(x)$ has a local maximum or minimum at $x = c$, then either $f'(c) = 0$ or f is not differentiable at c.

Values of c such that either $f'(c) = 0$ or f is not differentiable at c are called **critical values** for f.

Notice that the Extreme Value Theorem is talking about absolute maxima or minima, and Fermat's Theorem is talking about local maxima or minima (which include absolute maxima or minima but also include more).

Putting these together:

Finding the absolute maximum or absolute minimum of a nonconstant function $f(x)$ which is continuous on a closed interval $[a, b]$:

- Find the derivative $f'(x)$
- Find the values of x in $[a, b]$ for which $f'(x) = 0$ (critical values of the first type)
- Find the values of x in $[a, b]$ where the derivative does not exist (critical values of the second type)
- Evaluate $f(x)$ at each of those critical values
- Find the values of f at the endpoints: $f(a)$ and $f(b)$
- The largest of those values of f is its absolute maximum on the interval and the smallest of those values of f is its absolute minimum on the interval

The First and Second Derivative Tests for Local Maximum or Minimum

Recall that the first derivative is a function which gives the slope of the function $f(x)$ at a point on its graph.
If the slope is positive, the function is increasing. (The graph is going upward.)
If the slope is negative, the function is decreasing. (The graph is going downward.)
If there is a local maximum at $x = c$, the function must be increasing on the left of c and decreasing on the right of c. Therefore $f'(x)$ must be positive for $x < c$ and negative for $x > c$ (in some neighborhood of c)
If there is a local minimum at $x = c$, the function must be decreasing on the left of c and increasing on the right of c. Therefore $f'(x)$ must be negative for $x < c$ and positive for $x > c$ (in some neighborhood of c)

Putting these together we get:

The First Derivative Test for a Local Maximum or Minumum

- Find the critical values for $f(x)$
- The critical values (and the x-values where $f(x)$ is undefined, if there are any) divide the real line up into subintervals
- Choose an x-value in each subinterval and evaluate $f'(x)$ at that point. Note whether $f'(x)$ is positive or negative.
- If a critical value $x = c$ has $f'(x)$ positive to its left and $f'(x)$ negative to its right, there is a local maximum there.
- If a critical value $x = c$ has $f'(x)$ negative to its left and $f'(x)$ positive to its right, there is a local minimum there.
- If the sign of $f'(x)$ is the same on both sides of $x = c$, there is neither a local maximum nor a local minimum. Possibly it is a point of inflection (which we will discuss later).

The Second Derivative Test for a Local Maximum or Minumum

- Find the critical values for $f(x)$
- Find the second derivative $f''(x)$
- Evaluate $f''(x)$ at each of the critical values $x = c$
- If $f''(c)$ is negative, the graph is concave down, so there is a local maximum there.
- If $f''(c)$ is positive, the graph is concave up, so there is a local minimum there.
- If $f''(c) = 0$, no conclusion can be drawn from this test. We must use the First Derivative Test instead.