MAT 1475 Test 2

1. (*15 points*) Using the **definition of the derivative**, find the derivative of the function $f(x) = x^2 - 3x$

$$f(x+h) = (x+h)^2 - 3(x+h)$$
$$= x^2 + 2xh + h^2 - 3x - 3h$$

$$f(x+h) - f(x) = x^2 + 2xh + h^2 - 3x - 3h - (x^2 - 3x)$$
$$= \cancel{x^2} + 2xh + h^2 - \cancel{3x} - 3h - \cancel{x^2} + \cancel{3x}$$
$$= 2xh + h^2 - 3h$$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{2xh + h^2 - 3h}{h}$$
$$= \lim_{h \to 0} \frac{h(2x + h - 3)}{h}$$
$$= \lim_{h \to 0} (2x + h - 3) = 2x - 3$$

2. (*15 points*) Find the equation of the tangent line to the graph of $y = x^2 - 3x$ at the point $(2, -2)$. Note: this is the same function as in problem 1. You do not need to re-compute the derivative.

$$y' = 2x - 3$$
$$x = 2 \Rightarrow y' = 2(2) - 3 = 1 = m$$

tangent line: $y = mx + b$

$$y = 1 \cdot x + b = x + b$$

Substitute in $(2, -2)$

$$-2 = 2 + b$$
$$b = -4$$

$$\boxed{y = x - 4}$$

3. (*20 points*) Find y' for each of these functions using the derivative rules: simplify your answers.

a. (*10 pts*) $y = \dfrac{5x-2}{x^2+1}$

$$y' = \frac{(x^2+1)(5) - (5x-2)(2x)}{(x^2+1)^2}$$

$$= \frac{5x^2+5-10x^2+4x}{(x^2+1)^2}$$

$$= \boxed{\frac{-5x^2+4x+5}{(x^2+1)^2}}$$

b. (*10 pts*) $y = \sqrt{4-x^2}$

$$y = (4-x^2)^{1/2}$$

$$y' = \frac{1}{\cancel{2}}(4-x^2)^{-1/2} \cdot (-\cancel{2}x)$$

$$= \boxed{\frac{-x}{\sqrt{4-x^2}}}$$

4. (*15 points*) Find and simplify $f''(x)$ for the function $f(x) = \cos e^{2x}$

$$f'(x) = -\sin(e^{2x}) \cdot 2e^{2x} = -2e^{2x}\sin(e^{2x})$$

$$f''(x) = -4e^{2x}\sin(e^{2x}) - 2e^{2x}\sin(e^{2x}) \cdot 2e^{2x}$$

$$\boxed{\begin{aligned} &= -4e^{2x}\sin(e^{2x}) - 4e^{4x}\sin(e^{2x}) \\ &= -4\sin(e^{2x})\left[e^{2x} - e^{4x}\right] \end{aligned}}$$

5. (*20 points*) Use implicit differentiation to find $\frac{dy}{dx}$:

$$2x^2 - xy + y = 3$$

$$\frac{d}{dx}(2x^2) - \frac{d}{dx}(xy) + \frac{d}{dx}(y) = \frac{d}{dx}(3)$$

$$4x - y - x\frac{dy}{dx} + \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} - x\frac{dy}{dx} = y - 4x$$

$$\frac{dy}{dx}(1-x) = y - 4x$$

$$\boxed{\frac{dy}{dx} = \frac{y-4x}{1-x}}$$

6. (*15 points*) Use logarithmic differentiation to find $\frac{dy}{dx}$:

$$y = x^{\tan(x)}$$

$$\ln(y) = \ln(x^{\tan(x)})$$

$$\ln(y) = \tan(x) \cdot \ln(x)$$

$$\frac{d}{dx}(\ln(y)) = \frac{d}{dx}(\tan(x) \cdot \ln(x))$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \sec^2(x) \cdot \ln(x) + \tan(x) \cdot \frac{1}{x}$$

$$\frac{dy}{dx} = y\left(\sec^2(x) \cdot \ln(x) + \frac{\tan(x)}{x}\right)$$

$$\boxed{\frac{dy}{dx} = x^{\tan(x)}\left[\sec^2(x) \cdot \ln(x) + \frac{\tan(x)}{x}\right]}$$