Important note: in this course, all functions are considered as real-valued functions of a real variable.

Notation for limits

Limit of a function:
$\lim_{x \to c} f(x) = L$ means:

One-sided limits:
$\lim_{x \to c^+} f(x) = L$ means:

$\lim_{x \to c^-} f(x) = L$ means:

Theorem: If both one-sided limits $\lim_{x \to c^-} f(x)$ and $\lim_{x \to c^+} f(x)$ exist and are equal to the same number L, then $\lim_{x \to c} f(x) = L$ (the limit exists and is equal to that number).

Find an example of each of the following:

- A function which has a value c such that $f(a)$ is undefined, but $\lim_{x \to a} f(x) = L$ exists
- A function which has a value of c for which the two one-sided limits both exist, but they are not equal to each other.
- A function which has a value of c for which one of the one-sided limits exist but the other does not exist.

Infinite Limits:

$\lim_{x \to c} f(x) = \infty$ means:

$\lim_{x \to c} f(x) = -\infty$ means:

One-sided limits:
$\lim_{x \to c^+} f(x) = \infty$ means:

$\lim_{x \to c^+} f(x) = -\infty$ means:

$\lim_{x \to c^-} f(x) = \infty$ means:

$\lim_{x \to c^-} f(x) = -\infty$ means:

Note: Saying that a limit is equal to ∞ or $-\infty$ does not means that the limit exists! It is a special way that the limit fails to exist.

Limit properties

The limit properties are theorems about limits of functions, which can be used to simplify the computation of some limits.

Suppose that b is a constant and that both the limits $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist. Then:

- $\lim_{x\to c}[f(x)+g(x)] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$
  The limit of a sum is the sum of the limits.
- $\lim_{x\to c}[f(x)-g(x)] = \lim_{x\to c} f(x) - \lim_{x\to c} g(x)$
  The limit of a difference is the difference of the limits.
- $\lim_{x\to c} bf(x) = b\lim_{x\to c} f(x)$
  The limit of a constant times a function is the constant times the limit.
- $\lim_{x\to c}[f(x)g(x)] = \lim_{x\to c} f(x) \cdot \lim_{x\to a} g(x)$
  The limit of a product is the product of the limits.
- $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$ if $\lim_{x\to c} g(x) \neq 0$
  The limit of a quotient is the quotient of the limits, as long as the limit of the denominator is not zero.

More properties of limits that follow from the above:

- If $\lim_{x\to c} f(x) = L$, then $\lim_{x\to c}[f(x)]^n = L^n$
  The limit of a power is the power of the limits. Here n must be a positive integer.
- $\lim_{x\to c} b = b$
  The limit of a constant is the constant.
- $\lim_{x\to c} x = c$
- $\lim_{x\to c} x^n = c^n$
- $\lim_{x\to c} \sqrt[n]{x} = \sqrt[n]{c}$
  Here n is a positive integer, and if n is even we need to have $c > 0$
- If $\lim_{x\to c} f(x) = L$, then $\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{L}$
  Here n is a positive integer, and if n is even we need to have $\lim_{x\to c} f(x) > 0$

The substitution property:

If $f(x)$ is a polynomial function or a rational function and c is in the domain of f, then the limit of f as x approaches a can be found by substitution:
$\lim_{x\to c} f(x) = f(c)$

Warning: not all limits can be found by substitution!

Can you find an example of a limit which cannot be found by substitution?