In all the examples, I use the same strategy to organize my computations:
First compute $f(x+h)$, then compute $f(x+h)-f(x)$, and finally compute the derivative $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$.

Example: Find the derivative $f'(x)$ for $f(x)=\frac{1}{x+1}$

First we will compute $f(x+h)$:
$f(x+h)=\frac{1}{(x+h)+1}=\frac{1}{x+h+1}$
Note: the parentheses in the second expression are not needed: I only put them in there to show you that I was substituting $x+h$ in place of x.

Now we compute $f(x+h)-f(x)$:
$f(x+h)-f(x)=\frac{1}{x+h+1}-\frac{1}{x+1}$
Combine those rational expressions over the common denominator $(x+h+1)(x+1)$. Note: when we do that, we need to simplify the numerator, but **do not multiply out the denominator**, which just wastes time.

$$\frac{1}{x+h+1}-\frac{1}{x+1}=\frac{x+1}{(x+h+1)(x+1)}-\frac{x+h+1}{(x+h+1)(x+1)}$$

$$=\frac{(x+1)-(x+h+1)}{(x+h+1)(x+1)}$$

$$=\frac{x+1-x-h-1}{(x+h+1)(x+1)}$$

$$=\frac{-h}{(x+h+1)(x+1)}$$

So $f(x+h)-f(x)=\frac{-h}{(x+h+1)(x+1)}$

Now we will compute the derivative:

$$\begin{aligned}\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0}\frac{\left[\frac{-h}{(x+h+1)(x+1)}\right]}{h}\\ &= \lim_{h\to 0}\left[\frac{-h}{(x+h+1)(x+1)}\right]\frac{1}{h}\\ &= \lim_{h\to 0}\frac{-1}{(x+h+1)(x+1)}\\ &= \frac{-1}{(x+0+1)(x+1)}\\ &= \frac{-1}{(x+1)(x+1)}=\frac{-1}{(x+1)^2}\end{aligned}$$

So $f'(x)=\frac{-1}{(x+1)^2}$

Example: Find the derivative $f'(x)$ for $f(x) = |x|$

The absolute value function is a piecewise-defined function:
$|x| = \begin{cases} -x & x \leq 0 \\ x & x > 0 \end{cases}$ Therefore we will need to compute the limit that defines the derivative separately for the two intervals $(-\infty, 0)$ and $(0, \infty)$, and then one more time using one-sided limits for $x = 0$.

For $x < 0$, $f(x) = -x$, so $f(x+h) = -(x+h) = -x - h$

$f(x+h) - f(x) = (-x - h) - (-x) = -x - h + x = -h$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{-h}{h} = \lim_{h \to 0} (-1) = -1$$

So for x in $(-\infty, 0)$, $f'(x) = -1$

For $x > 0$, $f(x) = x$, so $f(x+h) = x + h$

$f(x+h) - f(x) = (x + h) - (-x) = h$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{h}{h} = \lim_{h \to 0} (1) = 1$$

So for x in $(0, \infty)$, $f'(x) = 1$

What happens when $x = 0$? Here we have to be careful because the absolute value function is defined differently on the left and on the right of 0, so we have to use a different formula in the difference quotient on the left and on the right. Therefore we find the two one-sided limits $\lim_{h \to 0^-} \frac{f(x+h) - f(x)}{h}$ and $\lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h}$ with $x = 0$, and see if they exist and are the same.

For $x < 0$, $f(x) = -x$, so $f(0 + h) = f(h) = -h$. [The 0 there is because we are doing this only for $x = 0$.]
Also $f(0) = 0$.

So $f(0 + h) - f(0) = -h - 0 = -h$

$$\lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^-} \frac{-h)}{h} = -1$$

For $x > 0$, $f(x) = x$, so $f(0 + h) = f(h) = h$. [Again, the 0 there is because we are doing this only for $x = 0$.]
Also $f(0) = 0$.

So $f(0 + h) - f(0) = h - 0 = h$

$$\lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^-} \frac{h)}{h} = 1$$

The two one-sided limits do not agree, so $f'(0)$ does not exist. The absolute value function is continuous everywhere on the real line, but it is differentiable only when $x \neq 0$.