Session 11: Rational Functions (summary of graphing)

First factor the numerator and denominator completely. Use the factored form to analyze the function **algebraically** to find the following:

- The domain of the function
  For a rational function, the domain is all real numbers except the x-values where the denominator of the function is 0.

- vertical asymptotes - see also the box below
  A vertical asymptote will occur at an x-value x=c where the denominator of the function is 0, with the factor (x-c) having multiplicity higher in the denominator than the multiplicity of that factor in the numerator: most commonly, when the denominator is 0 at x=c and the numerator is not 0 at x=c.

- holes (removable singularities) - see also the box below
  A hole will occur at an x-value x=c where the numerator of the function and the denominator are both 0, and the multiplicity of the factor (x-c) in the numerator is the same as or greater than the multiplicity in the denominator: most commonly, when both numerator and denominator have a single factor of (x-c).

- x- and y-intercepts
  The x-intercept(s) (if there are any) are found by setting f(x) = 0: for a rational function, this means that the numerator is equal to 0. Be careful to eliminate x-values which are not in the domain of the function.
  The y-intercept (if it exists) is found by setting x=0, namely, it is f(0), if 0 is in the domain of the function.

- Then determine the horizontal asymptote (if there is one) by looking at the behavior when $|x|$ is large. Remember that you only need to consider the leading terms of the numerator and denominator to find the end behavior of a rational function:
  If degree(numerator) > degree(denominator) - there is no horizontal asymptote.
  If degree(numerator) = degree(denominator) - the horizontal asymptote has the equation
  $$y = \text{the ratio of the leading coefficients.}$$
  If degree(numerator) < degree(denominator) - the horizontal asymptote has the equation y = 0 [the horizontal asymptote is the x-axis]

Finally, (only after you have determined these things analytically!), use the above information to select an appropriate viewing window or windows, and to interpret the calculator's version of the graph, so as to draw an accurate graph.

A useful trick to determine whether there is a vertical asymptote or a hole in the graph at a value $x = c$ where the denominator is 0:
Factor the numerator and denominator completely, then cancel any common factors of (x-c) between the numerator and the denominator.

- If after you cancel, the factor (x-c) is only in the denominator, there is a vertical asymptote at $x = c$.

- If after you cancel, there are no more factors of (x-c) in the denominator, there is a hole where $x = c$ on the graph of the function.

Example: $f(x) = \frac{x-1}{x^2-x-6}$

First analyze the function algebraically: factor the numerator and denominator completely: notice that there are no common factors between the numerator and the denominator, so we can use the shortcut in the second box above.

$$f(x) = \frac{x-1}{(x+2)(x-3)}$$

Now we can easily see the features listed above:

- The domain of this function: all real numbers except -2 and 3: that is, $\Re \setminus \{-2, 3\}$

- Vertical asymptotes: occur when the denominator is 0 and the numerator is **not** 0. So there are vertical asymptotes $x = -2$ and $x = 3$. [Note: the asymptotes are **lines**, and they have equations. An asymptote is not a number. You must give the equation of the asymptote.]

- Holes: occur when both the numerator and the denominator are 0. This graph has no holes.

- x-intercepts: occur when $y = 0$, that is, $f(x) = 0$. A rational function is 0 when its numerator is 0 and the denominator is not 0: so $x - 1 = 0$, $x = 1$ is the only x-intercept.

- y-intercepts: occur when x=0, that is, the y-intercept is $f(0) = \frac{0-1}{(0+2)(0-3)} = \frac{1}{6}$ after simplifying.

Then, we look to the behavior of $f(x)$ when $|x|$ is large. In this case, the degree of the numerator is less than the degree of the denominator: see what happens:
$f(x) \approx \frac{x}{x^2} = \frac{1}{x}$ when $|x|$ is large, and $\frac{1}{x}$ approaches 0 as x approaches ∞ or $-\infty$, so the horizontal asymptote is the line $y = 0$ (the x-axis).

Finally, since all of the lines and points from the above analysis will show up in the standard viewing window, we can use it to check the calculator's version of the graph. Notice that the calculator does not correctly display the graph: you need to interpret what you see based on the information you got analytically.

Sketch the graph in the graph paper below: