1) Consider the polynomial $f(x) = x^3 + x^2 + 9x + 9$

a) Find the exact values of the roots of $f(x)$. You must explain or show here how you know that they are roots of f(x): you may look at the graph of $f(x)$ to help you narrow down the possibilities, but you must use some algebraic means to show that they are actually roots.

This is a third degree polynomial, so we know from the Fundamental Theorem of Algebra that it will have 3 roots in the complex numbers, counting with multiplicity. By looking at the graph, it appears that there is a root at $x = -1$ and there are no other real-number roots visible. We can verify that this is a root and find the remaining factor by using synthetic or long division:

$$\begin{array}{r|rrrr} & 1 & 1 & 9 & 9 \\ -1 & & -1 & 0 & -9 \\ \hline & 1 & 0 & 9 & \boxed{0} \end{array}$$

The quotient is $x^2 + 9$, and we can find the remaining two roots using the square root property:
$x^2 + 9 = 0$
$x^2 = -9$
$x = \pm\sqrt{-9} = \pm 3i$
So the roots are $-1, 3i, -3i$

b) Use the roots you found in part (a) to factor f completely.
$f(x) = (x + 1)(x - 3i)(x + 3i)$

2) Solve the inequality and write your answer in interval form:
$x^3 - 3x > -2$

To make it easier, we make one side 0: $x^3 - 3x + 2 > 0$ We solve the corresponding equation:
$x^3 - 3x + 2 = 0$
Since this is a third degree polynomial equation and cannot easily be factored by elementary methods, we will use the same strategy we used in problem (1) to find its roots. Looking at the graph of $f(x) = x^3 - 3x + 2$, we see that it appears to have roots at $x = -2$ and $x = 1$, and the root at $x = 1$ appears to be a double root. We can verify this by synthetic or long division:

$$\begin{array}{r|rrrr} & 1 & 0 & -3 & 2 \\ -2 & & -2 & 4 & -2 \\ \hline & 1 & -2 & 1 & \boxed{0} \end{array}$$

The quotient is $x^2 - 2x + 1 = (x - 1)^2$, so the root at $x = 1$ is indeed a double root.
These two roots of the corresponding equation divide the real line into three intervals: $(-\infty, -2)$, $(-2, 1)$, and $(1, \infty)$. Now we either use a test point in each of those three intervals, or look at the graph, to determine which of those intervals will be in the solution set.

You can see the graph in desmos: the parts of the graph where $x^3 - 3x + 2 > 0$ are colored in red.
`https://www.desmos.com/calculator/5jhqwi6vz1`

It turns out that the intervals $(-2, 1)$, and $(1, \infty)$ are in the solution set.
Finally, we consider the endpoints. Since this is a strict inequality (it does not include the equals sign), the endpoints will not be included.
So the solution set is $(-2, 1) \cup (1, \infty)$

3) For the rational function $f(x) = \frac{x^2-4x}{2x^2-18}$

a) Find the x- and y-intercepts It is useful to factor the numerator and denominator before we do anything else:
$\frac{x^2-4x}{2x^2-18} = \frac{x(x-4)}{2(x^2-9)} = \frac{x(x-4)}{2(x-3)(x+3)}$

To find the x-intercept, we set $f(x) = 0$:
$\frac{x(x-4)}{2(x-3)(x+3)} = 0$
The only way a rational function can equal 0 is if the numerator is 0 and the denominator is not 0, so we need to solve $x(x-4) = 0$
$x = 0$ or $x = 4$
Note that the denominator is not 0 at either of these x-values. So there are two x-intercepts: $(0, 0)$ and $(4, 0)$.

To find the y-intercept, we would set $x = 0$ in the formula for f. But we already know this gives $y = 0$ because $(0, 0)$ is an x-intercept, and that must be the only point on the graph with x=0. So the y-intercept is $(0, 0)$.

b) Find the equations of the vertical and horizontal asymptotes.

There will be a vertical asymptote when the denominator is 0, and the numerator is not 0. The denominator is 0 when $x = 3$ or $x = -3$, and the numerator is not 0 there, so the equations of the two vertical asymptotes are $x = 3$ and $x = -3$.

We find the equation of the horizontal asymptote by looking at the leading terms in the numerator and denominator (going back to the unfactored form of f):
As $|x| \to \infty$, $y = f(x) = \frac{x^2-4x}{2x^2-18}\ \frac{x^2}{2x^2} = \frac{1}{2}$
So the horizontal asymptote is $y = \frac{1}{2}$

c) Sketch the graph on the graph paper. Make sure that you show the intercepts and the asymptotes and sketch the complete graph as accurately as you can.

You see this graph in desmos: the asymptotes are shown as dashed lines in orange, and the intercepts are shown in purple and labeled.
`https://www.desmos.com/calculator/5iki6cyuhk`
Note: this function appeared in one of the problems on the review self-tests!

4) Solve the inequality: write your answer in interval form $\frac{x^2-4x}{2x^2-18} \geq 0$

Notice that the left-hand side of this inequality is just the function whose graph we sketched in the previous problem! So most of the work is already done for this problem. The roots of the numerator are 0 and 4, and the roots of the denominator are -3 and 3. These four numbers divide the real line into five intervals: $(-\infty, -3)$, $(-3, 0)$, $(0, 3)$, $(3, 4)$, and $(4, \infty)$. By using a test point in each interval, or by looking at the graph, we can see that only the intervals $(-\infty, -3)$, $(0, 3)$, and $(4, \infty)$ are in the solution set. Now consider the endpoints: since this inequality includes the equals sign, the endpoints where $x = 0$ and where $x = 4$ are included; the endpoints $x = -3$ and $x = 3$ are not included because they are not in the domain of the rational function.

So the solution set is $(-\infty, -3) \cup [0, 3) \cup [4, \infty)$

Note: this is very similar to one of the problems on the review self-tests!

5) Find the domain of the function, the equation of the asymptote, and the x- and y-intercepts and then sketch the graph of the function: if there is no x- intercept or y-intercept, write "none"
$f(x) = \ln(1-x) + 2$

To find the domain: we need the argument of the logarithm to be positive:
$1 - x > 0$
$1 > x \implies x < 1$
So the domain is the interval $(-\infty, 1)$

To find the equation of the asymptote: this is where the argument of the logarithm is equal to 0, so it is the line $x = 1$

To find the x-intercept, we set the function equal to 0:
$\ln(1-x) + 2 = 0$
$\ln(1-x) = -2$
Rewrite in exponential form: $1 - x = e^{-2}$
$-x = e^{-2} - 1$
Multiply both sides by -1: $x = 1 - e^{-2} \approx 0.865$
So the x-intercept is at $(1 - e^{-2}, 0) \approx (0.865, 0)$

To find the y-intercept, we set x=0 in the formula for f:
$f(0) = \ln(1-0) + 2 = \ln(1) + 2 = 0 + 2 = 2$
So the y-intercept is $(0, 2)$

You can see the graph in desmos: the vertical asymptote is shown as a dashed line in orange, and the intercepts are marked and labeled.
`https://www.desmos.com/calculator/vevebhd44z`