Definition; natural number exponents $b^n = \underbrace{b \cdot b \cdot \cdots \cdot b}_{\text{n factors}}$

[Note: $b^1 = b$, only one "factor".]

Everything else follows from this definition and the elementary properties of arithmetic. Whenever it says below "by definition" referring to a natural number exponent, it means that we are using this definition.

Product of powers of the same base:

$b^n \cdot b^m = b^{n+m}$

add the powers

"Proof":

$$\begin{array}{rcll} b^2 \cdot b^3 & = & (b \cdot b)(b \cdot b \cdot b) & \text{by definition} \\ & = & b^5 & \text{by definition - count the number of factors of b in the product} \end{array}$$

Quotient of powers of the same base:

$\frac{b^n}{b^m} = b^{n-m}$

subtract the powers - top minus bottom

"Proof":

$$\begin{array}{rcll} \frac{b^5}{b^2} & = & \frac{b \cdot b \cdot b \cdot b \cdot b}{b \cdot b} & \text{by definition} \\ & = & b \cdot b \cdot b & \text{by "canceling" the common factors in the numerator and denominator} \\ & = & b^3 & \text{by definition} \end{array}$$

Note: The following two definitions are designed so that the two rules above will work correctly even with non-natural number exponents. This is (partly) explained by the motivations given after the definitions.

Definition of 0 power:

For any real number b, $b \neq 0$,

$b^0 = 1$

Note: 0^0 is undefined; in fact it is indefinite.

Motivation:

If $b \neq 0$ then

$$\begin{array}{rcll} \frac{b^5}{b^5} & = & 1 & \text{by a division property} \\ \text{but } \frac{b^5}{b^5} & = & b^{5-5} = b^0 & \text{if we use the "quotient of powers of the same base" rule} \\ \text{So let } b^0 & = & 1 & \text{to make this consistent} \end{array}$$

Definition of negative integer powers:

For any real number b, $b \neq 0$, and any positive integer n,

$b^{-n} = \frac{1}{b^n}$

Note: 0^{-n} is undefined because it would put 0 in the denominator.

Motivation:

If $b \neq 0$ then

$$\begin{array}{rcll} \frac{b^2}{b^5} & = & \frac{b \cdot b}{b \cdot b \cdot b \cdot b \cdot b} & \text{by definition} \\ & = & \frac{1}{b \cdot b \cdot b} & \text{by canceling} \\ \text{but } \frac{b^2}{b^5} & = & b^{2-5} = b^{-3} & \text{if we use the "quotient of powers of the same base" rule} \\ \text{So let } b^{-3} & = & \frac{1}{b^3} & \text{to make this consistent} \end{array}$$

Power to a power

$(b^n)^m = b^{nm}$

multiply the powers

"Proof":

$$\begin{aligned} (b^2)^3 &= b^2 \cdot b^2 \cdot b^2 && \text{by definition} \\ &= b^6 && \text{by the "product of powers of the same base" rule} \end{aligned}$$

Product to a power

$(ab)^n = a^n b^n$

the power distributes over the product

"Proof":

$$\begin{aligned} (ab)^3 &= (ab)(ab)(ab) && \text{by definition} \\ &= a \cdot a \cdot a \cdot b \cdot b \cdot b && \text{by the properties of multiplication} \\ &= a^3 b^3 && \text{by definition} \end{aligned}$$

Quotient to a power

$(\frac{a}{b})^n = \frac{a^n}{b^n}$

the power distributes over the quotient

"Proof":

$$\begin{aligned} (\tfrac{a}{b})^3 &= (\tfrac{a}{b})(\tfrac{a}{b})(\tfrac{a}{b}) && \text{by definition} \\ &= \tfrac{a \cdot a \cdot a \cdot}{b \cdot b \cdot b} && \text{by multiplication} \\ &= \tfrac{a^3}{b^3} && \text{by definition} \end{aligned}$$

Negative exponents in the denominator

$\frac{1}{b^{-n}} = b^n$

a negative exponent in the denominator means the same as a positive exponent in the numerator.

Note: putting this together with the definition $b^{-n} = \frac{1}{b^n}$ means we can always turn negative exponents into positive ones!

Proof: (real proof this time, not just "proof by example")

$$\begin{aligned} \tfrac{1}{b^{-n}} &= \tfrac{1}{\frac{1}{b^n}} && \text{by definition of the negative exponent} \\ &= 1 \div \tfrac{1}{b^n} && \text{interpreting the line as division} \\ &= 1 \cdot \tfrac{b^n}{1} && \text{divide by multiplying by the reciprocal} \\ &= b^n && \text{after simplifying} \end{aligned}$$

Negative powers and reciprocals

$(\frac{a}{b})^{-1} = \frac{b}{a}$

The -1 power is just a way of telling you to take the reciprocal.

Also: $(\frac{a}{b})^{-n} = (\frac{b}{a})^n$

The $-n$ power is the n-th power of the reciprocal.

Proof: (real proofs again) - The proof of the second assertion is similar. Try it!

$$\begin{aligned} (\tfrac{a}{b})^{-1} &= \tfrac{a^{-1}}{b^{-1}} && \text{by the "quotient to a power" rule} \\ &= \tfrac{\frac{1}{a}}{\frac{1}{b}} && \text{by definition of the negative power} \\ &= \tfrac{1}{a} \div \tfrac{1}{b} && \text{interpreting the line as division} \\ &= \tfrac{1}{a} \cdot \tfrac{b}{1} && \text{divide by multiplying by the reciprocal} \\ &= \tfrac{b}{a} && \text{after simplifying} \end{aligned}$$