Definition; natural number exponents $b^n = \underbrace{b \cdot b \cdot \dots \cdot b}_{\text{n factors}}$

[Note: $b^1 = b$, only one "factor".]

Everything else follows from this definition and the elementary properties of arithmetic. Whenever it says below "by definition" referring to a natural number exponent, it means that we are using this definition.

Product of powers of the same base:

$b^n \cdot b^m = b^{n+m}$

add the powers

"Proof":

$$\begin{aligned} b^2 \cdot b^3 &= (b \cdot b)(b \cdot b \cdot b) && \text{by definition} \\ &= b^5 && \text{by definition - count the number of factors of b in the product} \end{aligned}$$

Quotient of powers of the same base:

$\frac{b^n}{b^m} = b^{n-m}$

subtract the powers - top minus bottom

"Proof":

$$\begin{aligned} \tfrac{b^5}{b^2} &= \tfrac{b \cdot b \cdot b \cdot b \cdot b}{b \cdot b} && \text{by definition} \\ &= b \cdot b \cdot b && \text{by "canceling" the common factors in the numerator and denominator} \\ &= b^3 && \text{by definition} \end{aligned}$$

Note: The following two definitions are designed so that the two rules above will work correctly even with non-natural number exponents. This is (partly) explained by the motivations given after the definitions.

Definition of 0 power:

For any real number b, $b \neq 0$,

$b^0 = 1$

Note: 0^0 is undefined; in fact it is indefinite.

Motivation:

If $b \neq 0$ then

$$\begin{aligned} \tfrac{b^5}{b^5} &= 1 && \text{by a division property} \\ \text{but } \tfrac{b^5}{b^5} &= b^{5-5} = b^0 && \text{if we use the "quotient of powers of the same base" rule} \\ \text{So let } b^0 &= 1 && \text{to make this consistent} \end{aligned}$$

Definition of negative integer powers:

For any real number b, $b \neq 0$, and any natural number n,

$b^{-n} = \frac{1}{b^n}$

Note: 0^{-n} is undefined because it would put 0 in the denominator.

Motivation:

If $b \neq 0$ then

$$\begin{aligned} \tfrac{b^2}{b^5} &= \tfrac{b \cdot b}{b \cdot b \cdot b \cdot b \cdot b} && \text{by definition} \\ &= \tfrac{1}{b \cdot b \cdot b} && \text{by canceling} \\ \text{but } \tfrac{b^2}{b^5} &= b^{2-5} = b^{-3} && \text{if we use the "quotient of powers of the same base" rule} \\ \text{So let } b^{-3} &= \tfrac{1}{b^3} && \text{to make this consistent} \end{aligned}$$

Power to a power

$(b^n)^m = b^{nm}$

multiply the powers

"Proof":

$$\begin{aligned} (b^2)^3 &= b^2 \cdot b^2 \cdot b^2 && \text{by definition} \\ &= b^6 && \text{by the "product of powers of the same base" rule} \end{aligned}$$

Product to a power

$(ab)^n = a^n b^n$

the power distributes over the product

"Proof":

$$\begin{aligned} (ab)^3 &= (ab)(ab)(ab) && \text{by definition} \\ &= a \cdot a \cdot a \cdot b \cdot b \cdot b && \text{by the properties of multiplication} \\ &= a^3 b^3 && \text{by definition} \end{aligned}$$

Quotient to a power

$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

the power distributes over the quotient

"Proof":

$$\begin{aligned} \left(\frac{a}{b}\right)^3 &= \left(\frac{a}{b}\right)\left(\frac{a}{b}\right)\left(\frac{a}{b}\right) && \text{by definition} \\ &= \frac{a \cdot a \cdot a \cdot}{b \cdot b \cdot b} && \text{by multiplication} \\ &= \frac{a^3}{b^3} && \text{by definition} \end{aligned}$$

Negative exponents in the denominator

$\frac{1}{b^{-n}} = b^n$

a negative exponent in the denominator means the same as a positive exponent in the numerator.

Note: putting this together with the definition $b^{-n} = \frac{1}{b^n}$ means we can always turn negative exponents into positive ones!

Proof: (real proof this time, not just "proof by example")

$$\begin{aligned} \frac{1}{b^{-n}} &= \frac{1}{\frac{1}{b^n}} && \text{by definition of the negative exponent} \\ &= 1 \div \frac{1}{b^n} && \text{interpreting the line as division} \\ &= 1 \cdot \frac{b^n}{1} && \text{divide by multiplying by the reciprocal} \\ &= b^n && \text{after simplifying} \end{aligned}$$

Negative powers and reciprocals

$\left(\frac{a}{b}\right)^{-1} = \frac{b}{a}$

The -1 power is just a way of telling you to take the reciprocal.

Also: $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$

The $-n$ power is the n-th power of the reciprocal.

Proof: (real proofs again) - The proof of the second assertion is similar. Try it!

$$\begin{aligned} \left(\frac{a}{b}\right)^{-1} &= \frac{a^{-1}}{b^{-1}} && \text{by the "quotient to a power" rule} \\ &= \frac{\frac{1}{a}}{\frac{1}{b}} && \text{by definition of the negative power} \\ &= \frac{1}{a} \div \frac{1}{b} && \text{interpreting the line as division} \\ &= \frac{1}{a} \cdot \frac{b}{1} && \text{divide by multiplying by the reciprocal} \\ &= \frac{b}{a} && \text{after simplifying} \end{aligned}$$

Definition of rational number powers with numerator 1:

For any real number b, and any natural number n,
$b^{1/n} = \sqrt[n]{b}$
If n is even we also require that $b > 0$ so the root is a real number.

Motivation:
We have previously seen that the perfect n-th powers of a variable have powers which are multiples of n, and taking the n-th root means dividing the powers by n, for example:
$\sqrt{x^8} = x^4$ because $\left(x^4\right)^2 = x^8$
So it we write the square root as the 1/2 power and use the "power to a power rule", we get

$$\sqrt{x^8} = \left(x^8\right)^{1/2} = x^{\frac{1}{2}\cdot 8} = x^4$$

So everything works out the way it should.

Definition of rational number powers:

For any real number b, and any natural numbers m and n,
$b^{m/n} = \sqrt[n]{b^m}$
or
$b^{m/n} = \left(\sqrt[n]{b}\right)^m$
If n is even we also require that $b > 0$ so the root is a real number.

It does not matter which order the root and power are taken in.