CHAPTER IV

SIMILAR TRIANGLES

4.1 PROPORTIONS

In our discussion of similar triangles the idea of a proportion will play an important role. In this section we will review the important properties of proportions.

A proportion is an equation which states that two fractions are equal. For example, $\frac{2}{6} = \frac{4}{12}$ is a proportion. We sometimes say "2 is to 6 as 4 is to 12." This is also written 2:6 = 4:12. The extremes of this proportion are the numbers 2 and 12 and the means are the numbers 6 and 4. Notice that the product of the means $6 \times 4 = 24$ is the same as the product of the extremes $2 \times 12 = 24$.

THEOREM 1. If $\frac{a}{b} = \frac{c}{d}$ then $ad = bc$. Conversely, if $ad = bc$ then $\frac{a}{b} = \frac{c}{d}$. (The product of the means is equal to the product of the extremes).

EXAMPLES:

$\frac{2}{6} = \frac{4}{12}$ and $2 \times 12 = 6 \times 4$ are both true.

$\frac{2}{3} = \frac{6}{9}$ and $2 \times 9 = 3 \times 6$ are both true.

$\frac{1}{4} = \frac{4}{12}$ and $1 \times 12 = 4 \times 4$ are both false.

Proof of THEOREM 1: If $\frac{a}{b} = \frac{c}{d}$, multiply both sides of the equation by bd:

$$\frac{a}{\not b}(\not b d) = \frac{c}{\not d}(b\not d)$$

We obtain ad = bc.

Conversely, if ad = bc, divide both sides of the equation by bd:

$$\frac{a\not d}{b\not d} = \frac{\not b c}{\not b d}$$

The result is $\frac{a}{b} = \frac{c}{d}$.

The following theorem shows that we can interchange the means or the extremes or both of them simultaneously and still have a valid proportion:

THEOREM 2. If one of the following is true then they are all true:

(1) $\frac{a}{b} = \frac{c}{d}$

(2) $\frac{a}{c} = \frac{b}{d}$

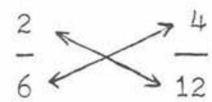
(3) $\frac{d}{b} = \frac{c}{a}$

(4) $\frac{d}{c} = \frac{b}{a}$

Proof: If any one of these proportions is true then ad = bc by THEOREM 1. The remaining proportions can then be obtained from ad = bc by division, as in THEOREM 1.

EXAMPLE: $\frac{2}{6} = \frac{4}{12}$, $\frac{2}{4} = \frac{6}{12}$, $\frac{12}{6} = \frac{4}{2}$, $\frac{12}{4} = \frac{6}{2}$ are all true because $2 \times 12 = 6 \times 4$.

The process of converting a proportion $\frac{2}{6} = \frac{4}{12}$ to the equivalent equation $2 \times 12 = 6 \times 4$ is sometimes called cross multiplication. The idea is conveyed by the following notation:



EXAMPLE A. Find x: $\frac{3}{x} = \frac{4}{20}$

Solution: By "cross multiplication,"

$$3(20) = x(4)$$
$$60 = 4x$$
$$15 = x$$

Check:

$\frac{3}{x} = \frac{3}{15} = \frac{1}{5}$. $\qquad$ $\frac{4}{20} = \frac{1}{5}$.

Answer: $x = 15$.

EXAMPLE B. Find x: $\frac{x - 1}{x - 3} = \frac{2x + 2}{x + 1}$

Solution:

$$(x - 1)(x + 1) = (x - 3)(2x + 2)$$
$$x^2 - 1 = 2x^2 - 4x - 6$$
$$0 = x^2 - 4x - 5$$
$$0 = (x - 5)(x + 1)$$
$$0 = x - 5 \qquad 0 = x + 1$$
$$5 = x \qquad -1 = x$$

Check, $x = 5$:

$$\frac{x - 1}{x - 3} = \frac{5 - 1}{5 - 3} = \frac{4}{2} = 2. \qquad \frac{2x + 2}{x + 1} = \frac{2(5) + 2}{5 + 1} = \frac{12}{6} = 2.$$

Check, $x = -1$:

$$\frac{x - 1}{x - 3} = \frac{-1 - 1}{-1 - 3} = \frac{-2}{-4} = \frac{1}{2}. \qquad \frac{2x + 2}{x + 1} = \frac{2(-1) + 2}{-1 + 1} = \frac{-2 + 2}{0} = \frac{0}{0}.$$

Since $\frac{0}{0}$ is undefined, we reject this answer.

Answer: $x = 5$.

PROBLEMS

1 - 12. Find x:

1. $\frac{6}{x} = \frac{18}{3}$

2. $\frac{4}{x} = \frac{2}{6}$

3. $\frac{x}{4} = \frac{9}{3}$

4. $\frac{x}{8} = \frac{9}{6}$

5. $\frac{7}{1} = \frac{x}{3}$

6. $\frac{10}{2} = \frac{25}{x}$

7. $\frac{x + 5}{x} = \frac{5}{4}$

8. $\frac{x - 6}{4} = \frac{5}{10}$

9. $\frac{3 + x}{x} = \frac{3}{2}$

10. $\frac{x}{x + 3} = \frac{4}{x}$

11. $\frac{3x - 3}{2x + 6} = \frac{x - 1}{x}$

12. $\frac{3x - 6}{x - 2} = \frac{2x + 2}{x - 1}$

4.2 SIMILAR TRIANGLES

Two triangles are said to be similar if they have equal sets of angles. In Figure 1, $\triangle ABC$ is similar to $\triangle DEF$. The angles which are equal are called corresponding angles. In Figure 1, $\angle A$ corresponds to $\angle D$, $\angle B$ corresponds to $\angle E$, and $\angle C$ corresponds to $\angle F$. The sides joining corresponding vertices are called corresponding sides. In Figure 1, AB corresponds to DE, BC corresponds to EF, and AC corresponds to DF. The symbol for similar is $\backsim$. The similarity statement $\triangle ABC \backsim \triangle DEF$ will always be written so that corresponding vertices appear in the same order. For the triangles in Figure 1, we could also write $\triangle BAC \backsim \triangle EDF$ or $\triangle ACB \backsim \triangle DFE$ but never $\triangle ABC \backsim \triangle EDF$ nor $\triangle ACB \backsim \triangle DEF$.

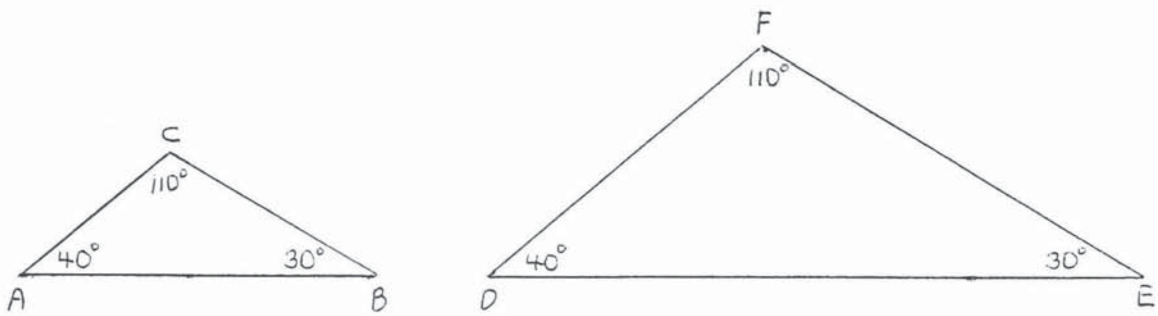
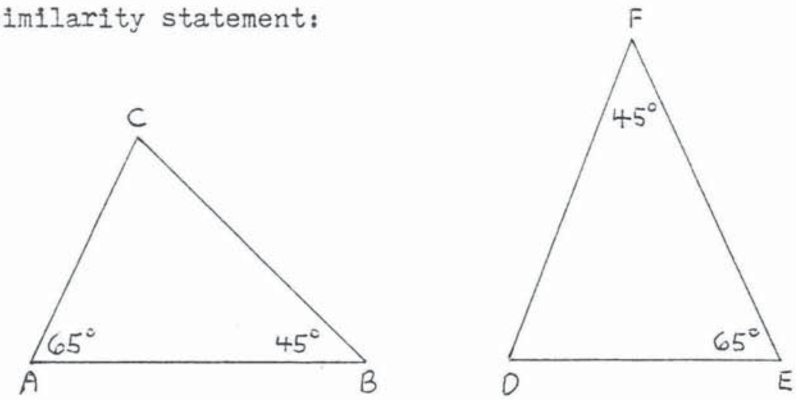


Figure 1. $\triangle ABC$ is similar to $\triangle DEF$.

We can tell which sides correspond from the similarity statement. For example, if $\triangle ABC \backsim \triangle DEF$, then side AB corresponds to side DE because both are the first two letters. BC corresponds to EF because both are the last two letters. AC corresponds to DF because both consist of the first and last letters.

EXAMPLE A. Determine if the triangles are similar, and if so, write the similarity statement:



Solution:

$\angle C = 180^\circ - (65^\circ + 45^\circ) = 180^\circ - 110^\circ = 70^\circ.$

$\angle D = 180^\circ - (65^\circ + 45^\circ) = 180^\circ - 110^\circ = 70^\circ.$

Therefore both triangles have the same angles and $\triangle ABC \sim \triangle EFD$.

Answer: $\triangle ABC \sim \triangle EFD$.

EXAMPLE A suggests that to prove similarity it is only necessary to know that two of the corresponding angles are equal:

THEOREM 1. Two triangles are similar if two angles of one equal two angles of the other (AA = AA).

In Figure 2, $\triangle ABC \sim \triangle DEF$ because $\angle A = \angle D$ and $\angle B = \angle E$.

Proof:

$\angle C = 180^\circ - (\angle A + \angle B) = 180^\circ - (\angle D + \angle E) = \angle F.$

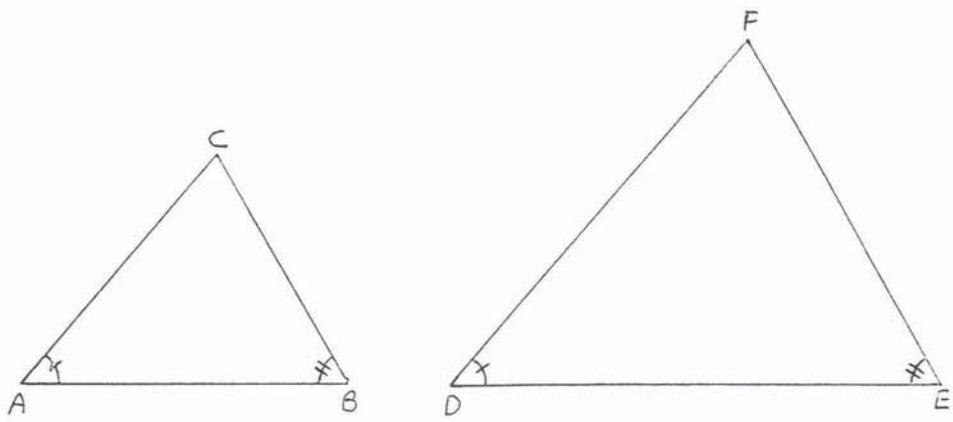
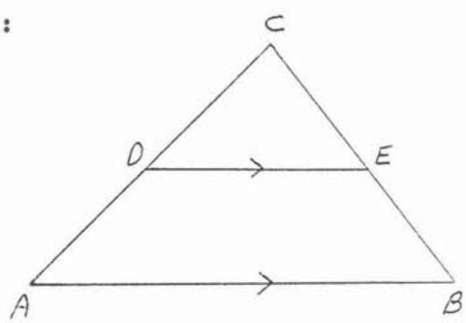


Figure 2. $\triangle ABC \sim \triangle DEF$ because AA = AA.

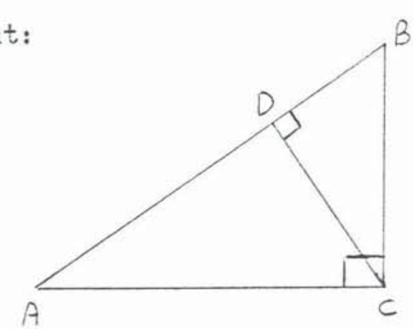
EXAMPLE B. Determine which triangles are similar and write a similarity statement:



Solution: $\angle A = \angle CDE$ because they are corresponding angles of parallel lines. $\angle C = \angle C$ because of identity. Therefore $\triangle ABC \sim \triangle DEC$ by AA = AA.

Answer: $\triangle ABC \sim \triangle DEC$.

EXAMPLE C. Determine which triangles are similar and write a similarity statement:



Solution: $\angle A = \angle A$, identity. $\angle ACB = \angle ADC = 90^\circ$. Therefore

$$\triangle ABC \sim \triangle ACD$$

$$90^\circ$$

Also $\angle B = \angle B$, identity. $\angle BDC = \angle BCA = 90^\circ$. Therefore

$$\triangle ABC \sim \triangle CBD$$

$$90^\circ$$

Answer: $\triangle ABC \sim \triangle ACD \sim \triangle CBD$.

Similar triangles are important because of the following theorem:

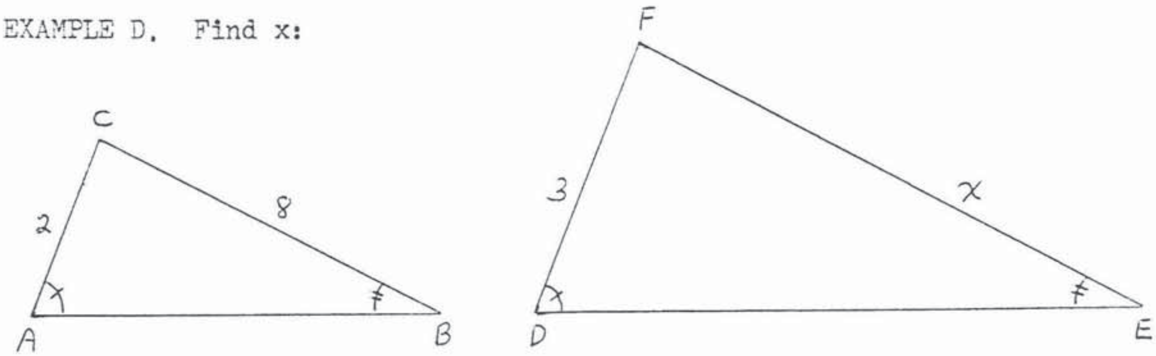
THEOREM 2. The corresponding sides of similar triangles are proportional. This means that if $\triangle ABC \sim \triangle DEF$ then

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}.$$

That is, the first two letters of $\triangle ABC$ are to the first two letters of $\triangle DEF$ as the last two letters of $\triangle ABC$ are to the last two letters of $\triangle DEF$ as the first and last letters of $\triangle ABC$ are to the first and last letters of $\triangle DEF$.

Before attempting to prove THEOREM 2, we will give several examples of how it is used:

EXAMPLE D. Find x:



Solution: $\angle A = \angle D$ and $\angle B = \angle E$ so $\triangle ABC \sim \triangle DEF$. By THEOREM 2,

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}.$$

We will ignore $\frac{AB}{DE}$ here since we do not know and do not have to find either AB or DE.

$$\frac{BC}{EF} = \frac{AC}{DF}$$

$$\frac{8}{x} = \frac{2}{3}$$

$$24 = 2x$$

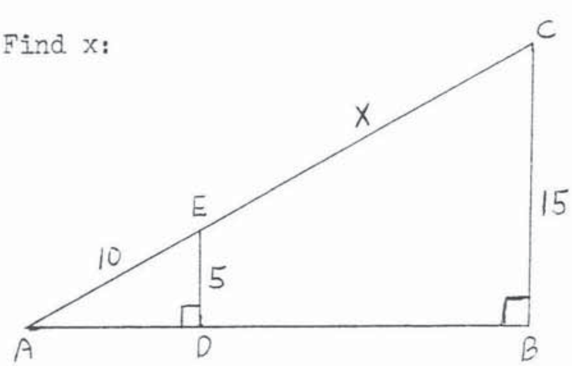
$$12 = x$$

Check:

$\frac{BC}{EF}$	$\frac{AC}{DF}$
$\frac{8}{x}$	$\frac{2}{3}$
$\frac{8}{12}$	
$\frac{2}{3}$	

Answer: x = 12.

EXAMPLE E. Find x:



Solution: $\angle A = \angle A$, $\angle ADE = \angle ABC$, so $\triangle ADE \sim \triangle ABC$ by AA = AA.

$$\frac{AD}{AB} = \frac{DE}{BC} = \frac{AE}{AC}.$$

We ignore $\frac{AD}{AB}$.

$$\frac{DE}{BC} = \frac{AE}{AC}$$

$$\frac{5}{15} = \frac{10}{10 + x}$$

$$5(10 + x) = 15(10)$$

$$50 + 5x = 150$$

$$5x = 150 - 50$$

$$5x = 100$$

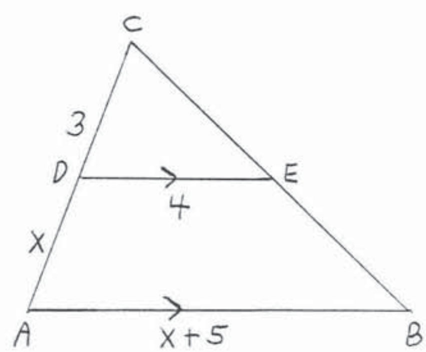
$$x = 20$$

Check: $\frac{DE}{BC} = \frac{AE}{AC}$

$\frac{5}{15}$	$\frac{10}{10 + x}$
$\frac{1}{3}$	$\frac{10}{10 + 20}$
	$\frac{10}{30}$
	$\frac{1}{3}$

Answer: $x = 20$.

EXAMPLE F. Find x:



Solution: $\angle A = \angle CDE$ because they are corresponding angles of parallel lines. $\angle C = \angle C$ because of identity. Therefore $\triangle ABC \sim \triangle DEC$ by AA = AA.

$$\frac{AB}{DE} = \frac{BC}{EC} = \frac{AC}{DC}$$

We ignore $\frac{BC}{EC}$:

$$\frac{AB}{DE} = \frac{AC}{DC}$$

$$\frac{x+5}{4} = \frac{x+3}{3}$$

$$(x+5)(3) = (4)(x+3)$$

$$3x + 15 = 4x + 12$$

$$15 - 12 = 4x - 3x$$

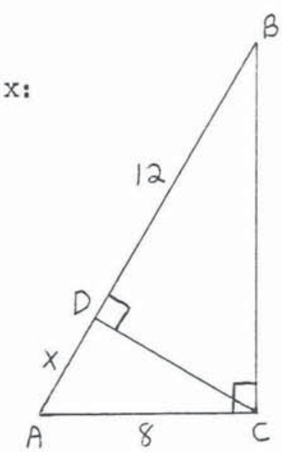
$$3 = x$$

Check: $\frac{AB}{DE} = \frac{AC}{DC}$

$\frac{x+5}{4}$	$\frac{x+3}{3}$
$\frac{3+5}{4}$	$\frac{3+3}{3}$
$\frac{8}{4}$	$\frac{6}{3}$
2	2

Answer: $x = 3$.

EXAMPLE G. Find x:



Solution: $\angle A = \angle A$, $\angle ACB = \angle ADC = 90^\circ$. $\triangle ABC \sim \triangle ACD$.

$$\frac{AB}{AC} = \frac{AC}{AD}$$

$$\frac{x+12}{8} = \frac{8}{x}$$

$$(x+12)(x) = (8)(8)$$

$$x^2 + 12x = 64$$

$$x^2 + 12x - 64 = 0$$

$$(x - 4)(x + 16) = 0$$

$$x = 4 \qquad x = -16$$

We reject the answer x = -16 because AD = x cannot be negative.

Check, x = 4:

$$\frac{AB}{AC} = \frac{AC}{AD}$$

$$\frac{x + 12}{8} \;\Big|\; \frac{8}{x}$$

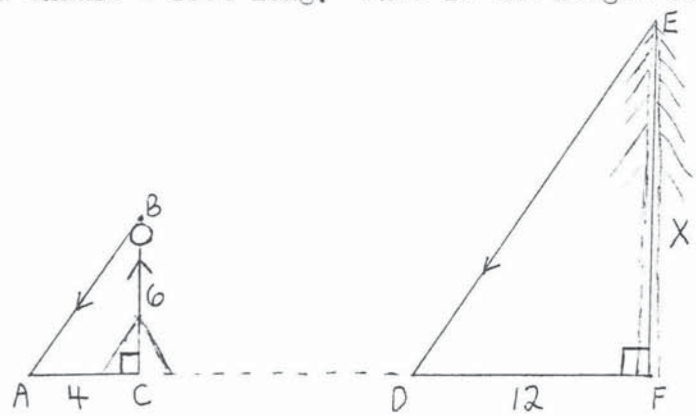
$$\frac{4 + 12}{8} \;\Big|\; \frac{8}{4}$$

$$\frac{16}{8} \;\Big|\; 2$$

$$2 \;\Big|$$

Answer: x = 4.

EXAMPLE H. A tree casts a shadow 12 feet long at the same time a 6 foot man casts a shadow 4 feet long. What is the height of the tree?



Solution: In the diagram AB and DE are parallel rays of the sun. Therefore $\angle A = \angle D$ because they are corresponding angles of parallel lines with respect to the transversal AF. Since also $\angle C = \angle F = 90^\circ$, we have $\triangle ABC \sim \triangle DEF$ by AA = AA.

$$\frac{AC}{DF} = \frac{BC}{EF}$$

$$\frac{4}{12} = \frac{6}{x}$$

$$4x = 72$$

$$x = 18$$

Answer: x = 18 feet.

Proof of THEOREM 2 ("The corresponding sides of similar triangles are proportional"):

We illustrate the proof using the triangles of EXAMPLE D (Figure 3). The proof for other similar triangles follows the same pattern. Here we will prove that x = 12 so that $\frac{2}{3} = \frac{8}{x}$.

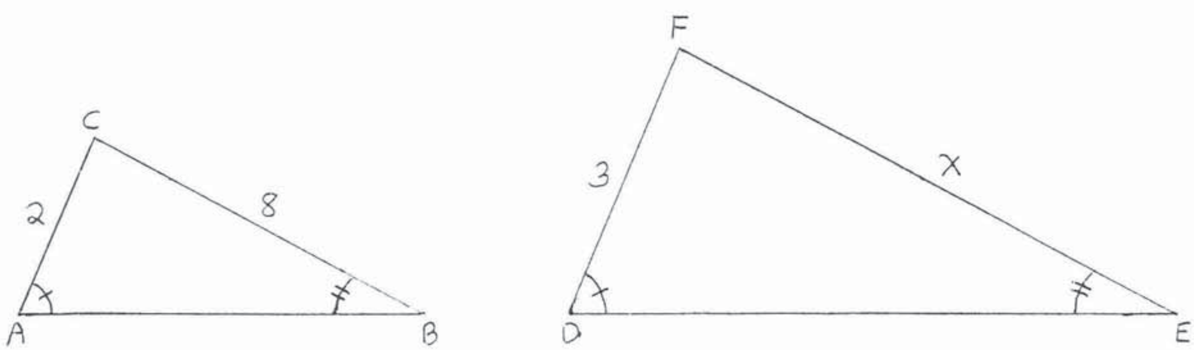


Figure 3. The triangles of EXAMPLE D.

First draw lines parallel to the sides of $\triangle ABC$ and $\triangle DEF$ as shown in Figure 4. The corresponding angles of these parallel lines are equal and each of the parallelograms with a side equal to 1 has its opposite side equal to 1 as well. Therefore all of the small triangles with a side equal to 1 are congruent by AAS = AAS. The corresponding sides of

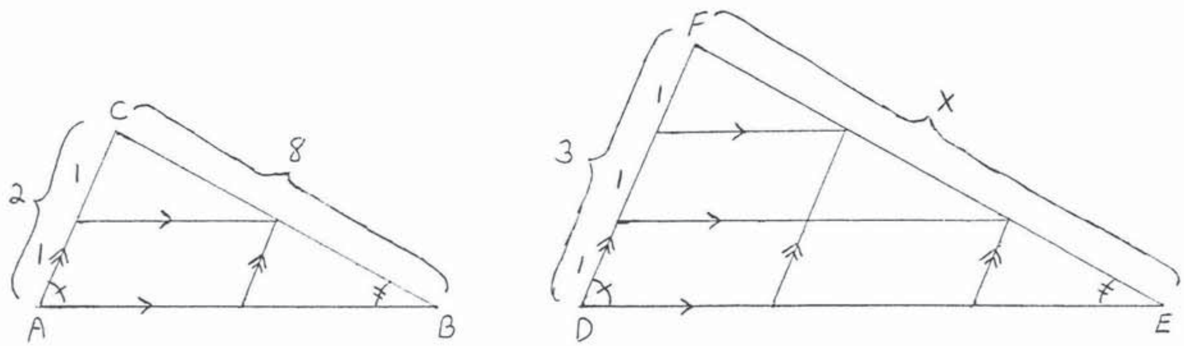
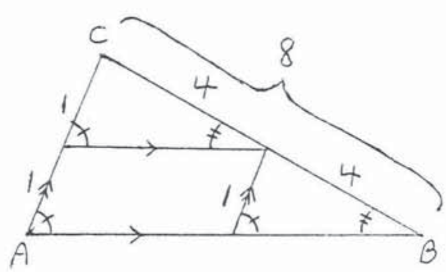


Figure 4. Draw lines parallel to the sides of $\triangle ABC$ and $\triangle DEF$.

these triangles form side BC = 8 of $\triangle ABC$ (see Figure 5). Therefore each of these sides must equal 4 and x = EF = 4 + 4 + 4 = 12 (Figure 6).



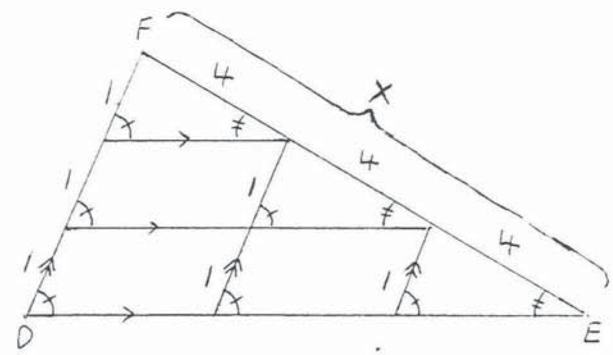


Figure 5. The small triangles are congruent hence the corresponding sides lying on BC must each be equal to 4.

Figure 6. The small triangles of $\triangle DEF$ are congruent to the small triangles of $\triangle ABC$ hence x = EF = 4 + 4 + 4 = 12.

(Note to instructor: This proof can be carried out whenever the lengths of the sides of the triangles are rational numbers. However, since irrational numbers can be approximated as closely as necessary by rationals, the proof extends to that case as well.)

Historical Note: Thales (c. 600 B.C.) used the proportionality of sides of similar triangles to measure the heights of the pyramids in Egypt. His method was much like the one we used in EXAMPLE H to measure the height of trees.

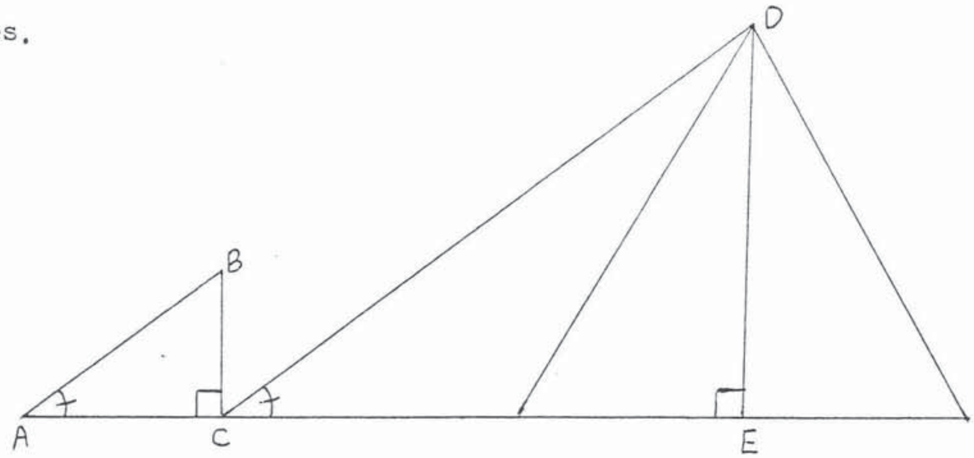


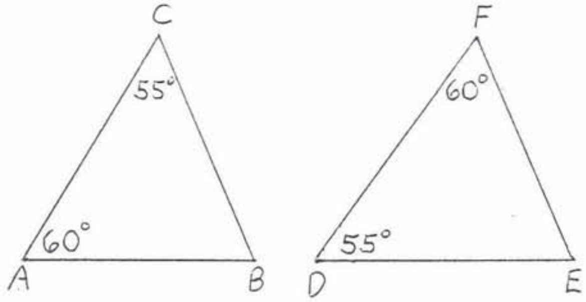
Figure 7. Using similar triangles to measure the height of a pyramid.

In Figure 7, DE represents the height of the pyramid and CE is the length of its shadow. BC represents a vertical stick and AC is the length of its shadow. We have $\triangle ABC \sim \triangle CDE$. Thales was able to measure directly the lengths AC, BC, and CE. Substituting these values in the proportion $\frac{BC}{DE} = \frac{AC}{CE}$, he was able to find the height DE.

PROBLEMS

1 - 6. Determine which triangles are similar and write the similarity statement:

1.



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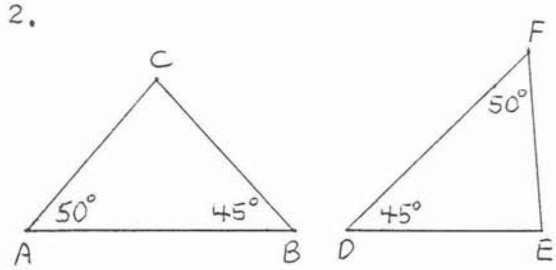

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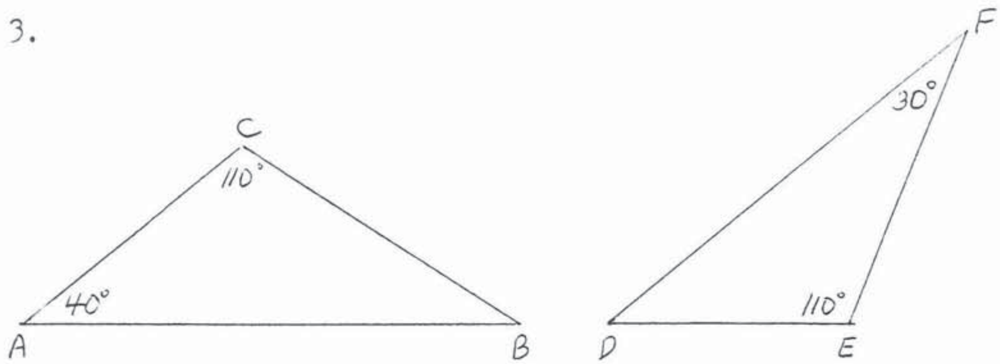

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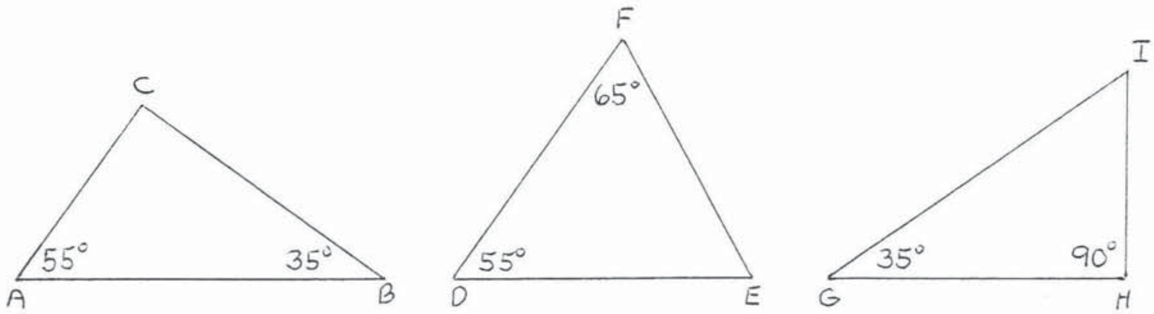

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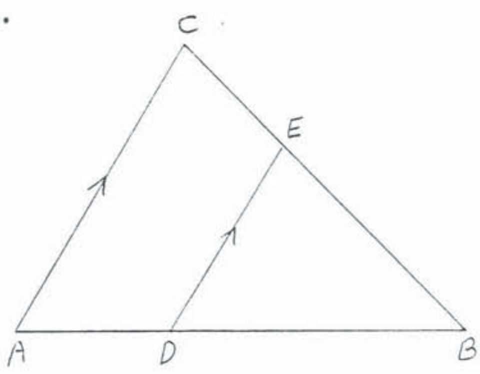

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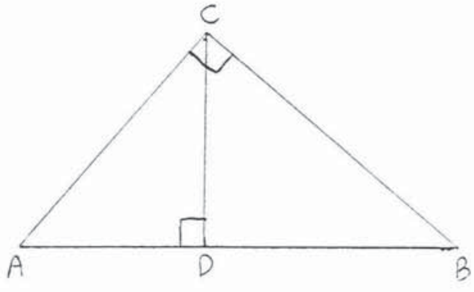
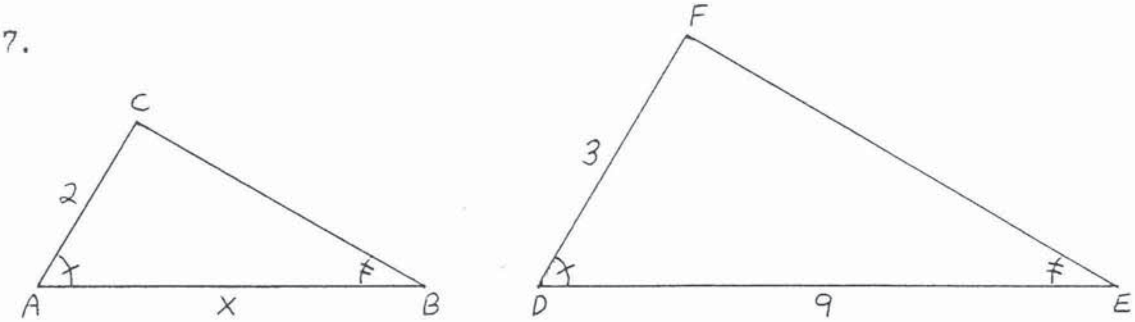


7 - 22. For each of the following (1) write the similarity statement, (2) write the proportion between the corresponding sides, and (3) solve for x or x and y.

7.



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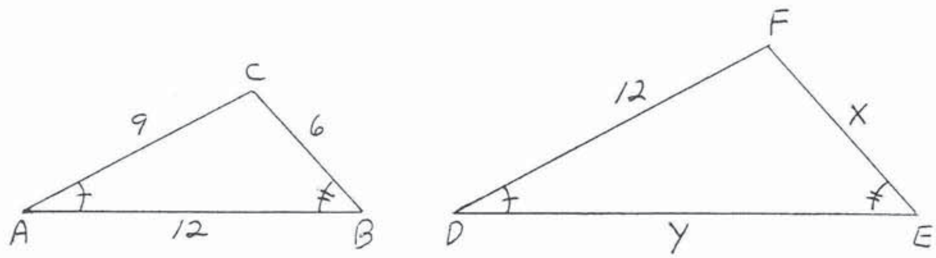

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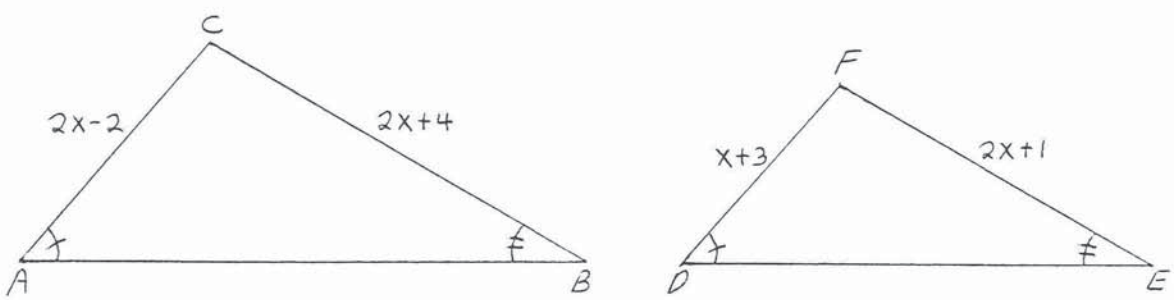

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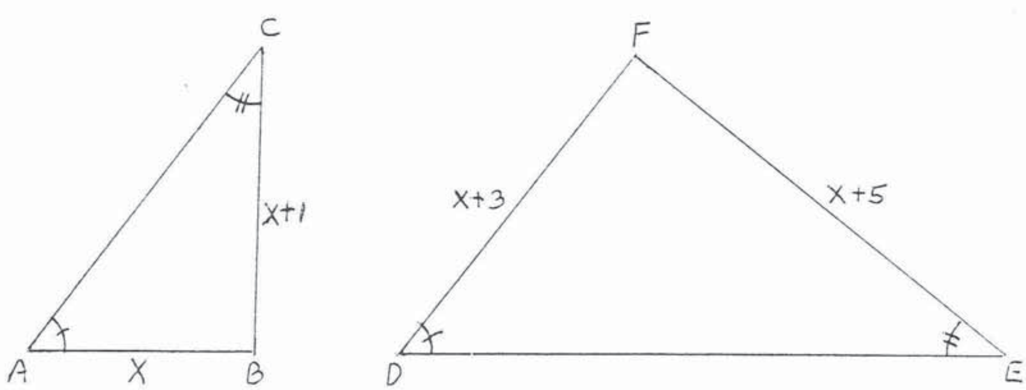

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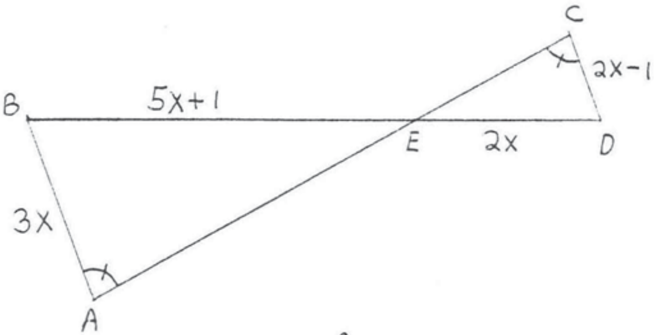

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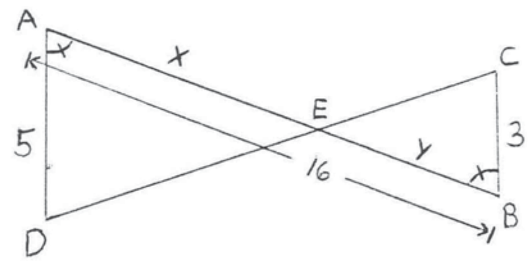

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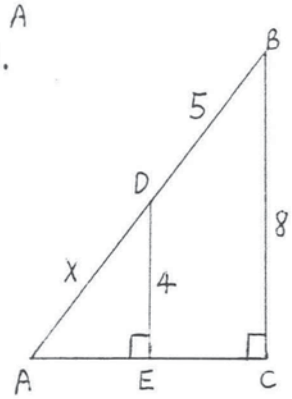

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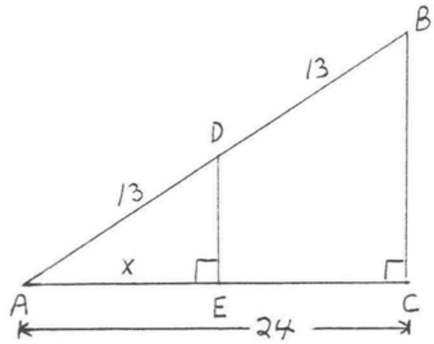

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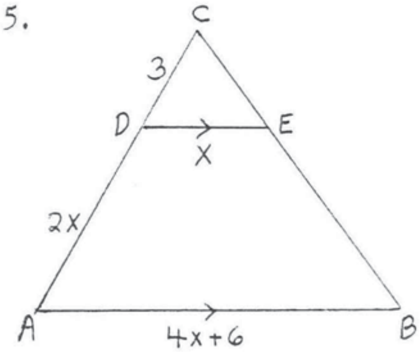

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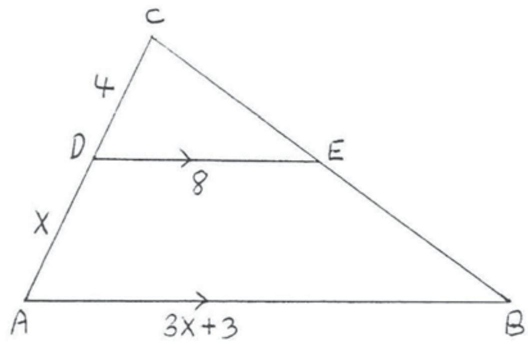

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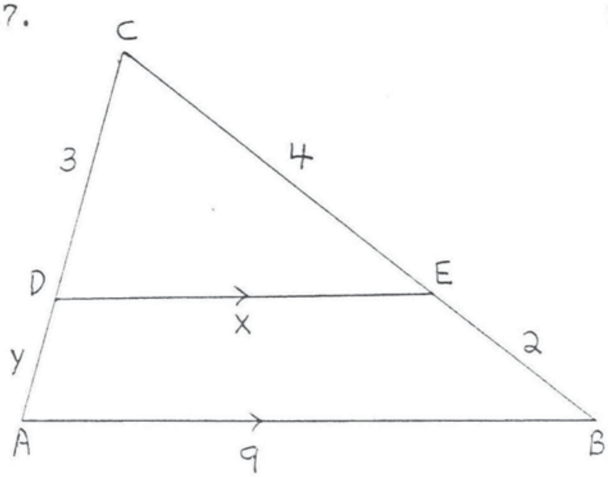

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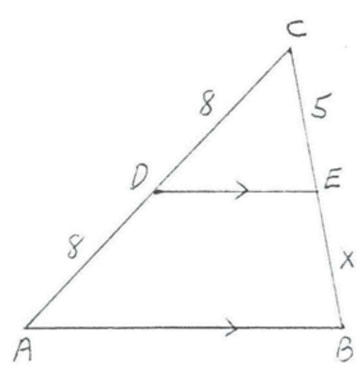

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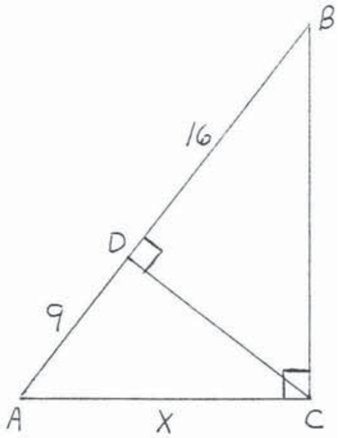

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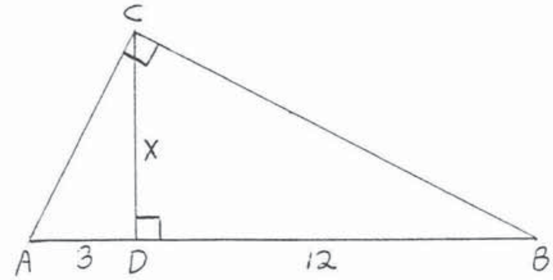

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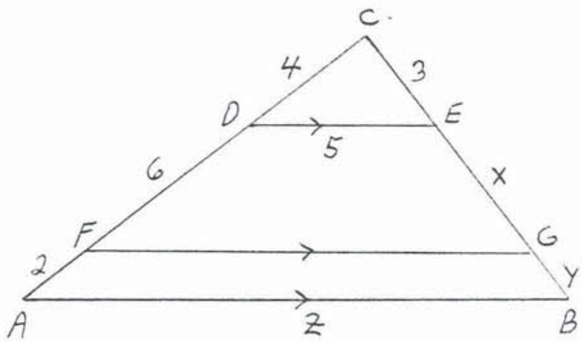
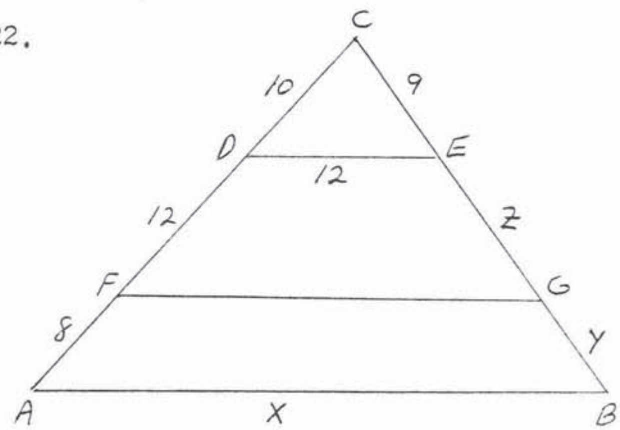


22.



23. A flagpole casts a shadow 80 feet long at the same time a 5 foot boy casts a shadow 4 feet long. How tall is the flagpole?

24. Find the width AB of the river: