

Lecture 12

Partial Differential Equations PDE

Just like ordinary differential equations (ODE), partial differential equations (PDE) are a way to express a function y through its derivatives. The goal is to find the actual values of y . What makes a PDE a PDE rather than an ODE is that it is supposed to be a function of more than one variable and that the derivatives are with respect to different variables. We discuss here two important types of PDEs, the wave equation and the heat equation:

Wave equation:

$$\frac{\partial^2}{\partial t^2} y(x, t) - c^2 \frac{\partial^2}{\partial x^2} y(x, t) = 0$$

Heat equation:

$$\frac{\partial}{\partial t} y(x, t) - D \frac{\partial^2}{\partial x^2} y(x, t) = 0$$

Heat and wave equations differ by the order of the time derivative. The space derivative above shows one dimensional space, however the equivalent multi-dimensional case just contains the additional second order terms for each dimension.

The actual solution of the problem depends on initial conditions. For PDE's those are called boundary conditions, as one needs to specify the function y not just for a point but for the full slices in space and time, for example starting $y(x, 0)$ value and boundary values $y(a, t)$ and $y(b, t)$ across all times.

Again we will solve the PDE's with Fourier methods. Let's look at a single frequency in time and space.

$$y(x, t) = e^{i(kx - \omega t)}$$

$$\frac{\partial}{\partial t} y(x, t) = y_t = -i\omega y$$

$$\frac{\partial^2}{\partial t^2} y(x, t) = y_{tt} = -\omega^2 y$$

$$\frac{\partial}{\partial x} y(x, t) = y_x = ik y$$

$$\frac{\partial^2}{\partial x^2} y(x, t) = y_{xx} = -k^2 y$$

Substituting into the respective PDE's we get

Wave equation:

$$-\omega^2 + c^2 k^2 = 0$$

$$c = \frac{\omega}{k}$$

$$y(x, t) = e^{ik(x-ct)}$$

Alternatively

$$y(x, t) = \cos(k(x - ct))$$

Verifying

```
syms x t k c w;  
y = cos(k*(x-c*t));  
dx = diff(y,x);  
dxx = diff(dx,x);  
dt = diff(y,t);  
dtt = diff(dt,t);  
dtt - c^2* dxx
```

```
ans = 0
```

Wave propagates with speed c

Heat equation:

$$-i\omega + Dk^2 = 0$$

$$i\omega = Dk^2$$

$$y(x, t) = e^{ikx - Dk^2 t}$$

Exponential decay with factor Dk^2

Alternatively

$$y(x, t) = e^{-Dk^2 t} \cos(kx)$$

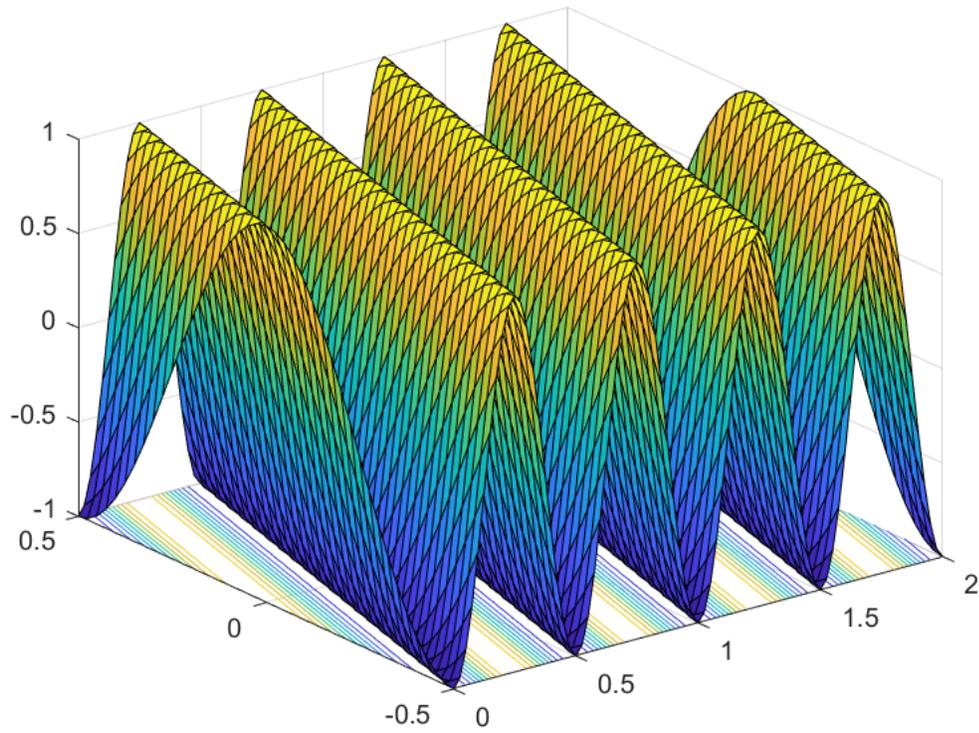
Verifying

```
syms x t k D;  
y = cos(k*x)*exp(-D*k^2*t);  
dx = diff(y,x);  
dxx = diff(dx,x);  
dt = diff(y,t);  
dt - D* dxx
```

```
ans = 0
```

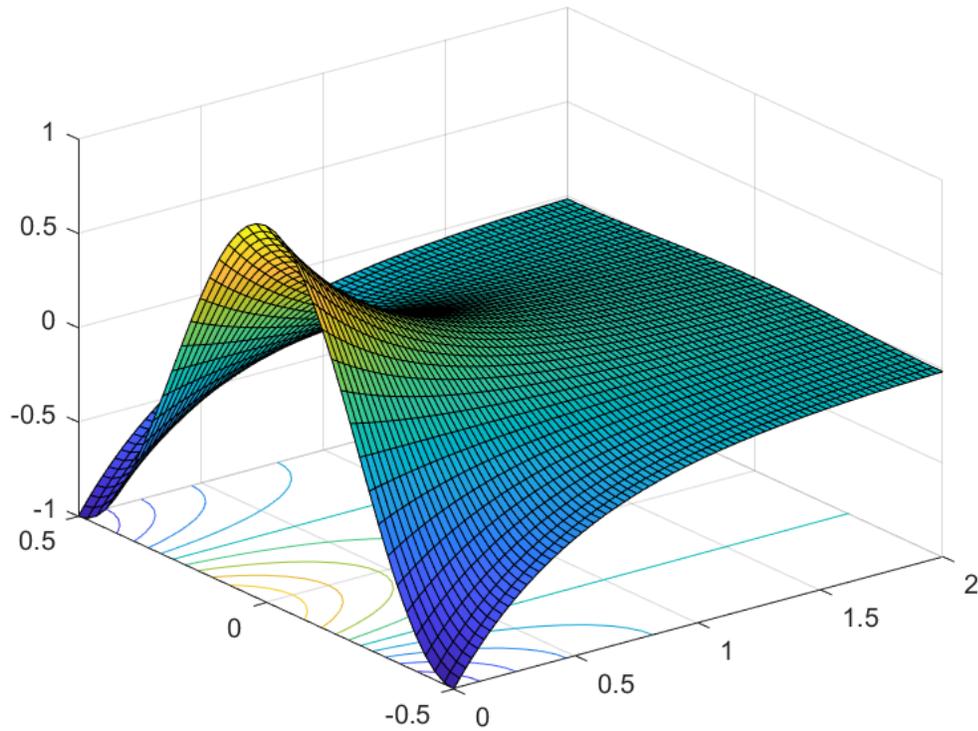
Plotting a traveling wave solution

```
k = 2*pi;  
c = 2;  
[t,x] = meshgrid(0:0.025:2, -0.5:0.025:0.5);  
y = cos(k*(x-c*t));  
surfc(t,x,y)
```



Plotting a heat diffusion solution

```
k = 2*pi;  
D = .05;  
[t,x] = meshgrid(0:0.025:2,-0.5:0.025:0.5);  
y = cos(k*x).*exp(-D.*(k.^2).*t);  
surfc(t,x,y)
```



Fourier Transformations

Above we found solutions for particular trig functions. In order to generate a more general solution that matches a particular initial condition we use Fourier transformations. For the heat equation we found the solution

$$y(x, t) = e^{ikx - Dk^2 t}$$

which is for time 0

$$y(x, 0) = e^{ikx}$$

This means that any initial condition that can be written at time zero as

$$y(x, 0) = \int u(k) e^{ikx} dk$$

generates a general solution of the form as

$$y(x, t) = \int u(k) e^{-Dk^2 t} e^{ikx} dk$$

If we specify $y(x, 0)$ than $u(k)$ will be its inverse Fourier transform. Then, taking the Fourier transform of $u(k)$ in combination with the decaying factor $e^{-Dk^2 t}$ provides us with the solution how the the initial function is propagated in time.

Lets look for the simple case of $y(x, 0) = \delta(x, 0)$, a initial spike delta function. It's inverse Fourier transform is

$u(k) = \frac{1}{2\pi}$. The solution for $y(x,t)$ over time becomes

```
syms D t k x
assume(D>0)
assume(t>0)
u = ifourier(dirac(x))
```

u =

$$\frac{1}{2\pi}$$

```
gauss = fourier(u* exp(-D*t*k^2),k,x)
```

gauss =

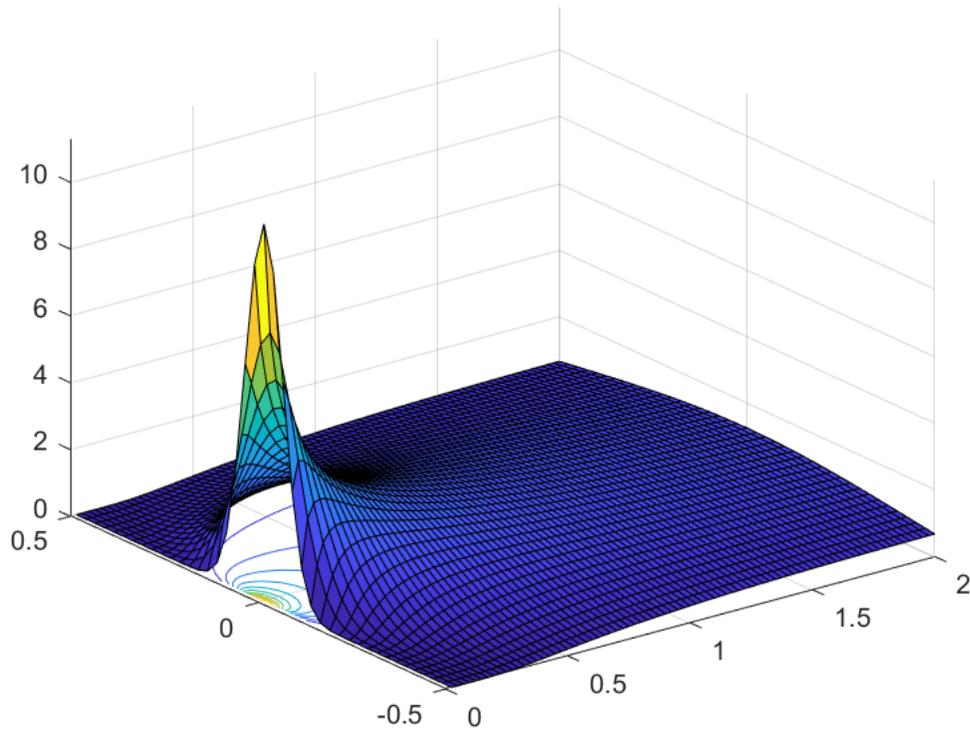
$$\frac{e^{-\frac{x^2}{4Dt}}}{2\sqrt{D}\sqrt{t}\sqrt{\pi}}$$

The resulting function is also known as a Gaussian normal distribution function.

$$y(x, t) = \frac{1}{\sqrt{2\pi}\sqrt{2Dt}} e^{-\frac{x^2}{4Dt}}$$

Let's plot the solution of the heat equation in the case of a point source (dirac delta function) at time 0.

```
k = 2*pi;
D = .05;
[t,x] = meshgrid(0:0.025:2,-0.5:0.025:0.5);
y = (2*pi*D*t).^(-0.5).*exp(-(x.^2).*(4*D.*t).^(-1));
surfc(t,x,y)
```



Note that we used the "." syntax above, for example ".*" and ".^" in order to indicate element-wise calculation not matrix multiplications of the meshgrid entries.

Numerical Fourier Transformations

It is not always possible to compute the inverse Fourier transform analytically. In this case we can go back to the fast fourier transform from previous lectures.

In the fast fourier transform, the function $y(x,0)$ is composed of terms

$$y(x_n, t) = \frac{1}{N} u(k) e^{\frac{2\pi i k n}{N}} f(t) = \frac{1}{N} u(k) e^{\frac{2\pi i k x}{L}} f(t) \text{ with } x = \frac{n}{N} L \text{ or } n = \frac{N}{L} x$$

Applying the heat PDE $\frac{\partial}{\partial t} y(x, t) - D \frac{\partial^2}{\partial x^2} y(x, t) = 0$ yields

$$f'(t) + 4\pi^2 \left(\frac{k}{L}\right)^2 D f(t) = 0 \text{ or } \frac{f'}{f} = \frac{d}{dt} \ln(f) = -4\pi^2 \left(\frac{k}{L}\right)^2 D$$

which has the solution

$$f(t) = e^{-4\pi^2 \left(\frac{k}{L}\right)^2 D t} \text{ or in total for each frequency } k$$

$$y(x_n, t) = \frac{1}{N} u(k) e^{-4\pi^2 \left(\frac{k}{L}\right)^2 D t} e^{2\pi i \frac{k}{N} n}$$

For example instead using a delta function as time=0 initial condition, let's use a step function that is

$$y(x, 0) = 1 \text{ for } -0.1 < x < 0.1 \text{ and } y(x, 0) = 0 \text{ otherwise}$$

In the code below we compute the frequencies $u(k)$ from the initial function y_0 , and then multiply by factors

$$e^{-4\pi^2 \left(\frac{k}{L}\right)^2 D t}$$

to generate solutions of $y(t, x)$ for different time slice.

Note that the the fast fourier transform base frequencies are mirrored, so that the k vector has the form

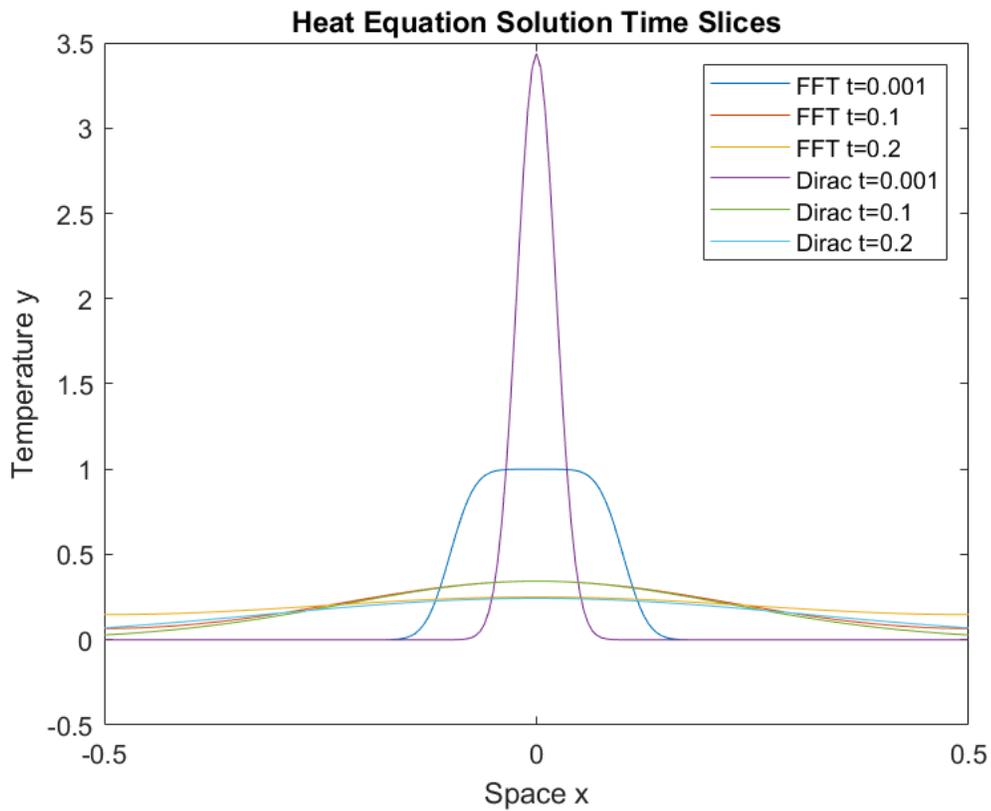
$$k = [0 \ 1 \ 2 \ \dots \ n_k \ n_k \ n_k - 1 \ n_k - 2 \ \dots \ 2 \ 1]$$

```

D=0.25; % diffusion constant
N = 201;
l = .5; % computation boundary from -l to l
L = 2*l;% total length
x= linspace(-l,l,N); % x-dimension
nk=((length(x))-1)/2; % midpoint frequency
y0=heaviside(x+0.1)-heaviside(x-0.1); % initial condition
u = ifft(y0); % inverse transform gives frequencies of initial condition
k= [0:nk nk:(-1):1]; % frequency vector
t=.001;
y1=real(fft(u.*exp(-4*pi^2*D*t*(k/L).^2)));
t=.1;
y2=real(fft(u.*exp(-4*pi^2*D*t*(k/L).^2)));
t=.2;
y3=real(fft(u.*exp(-4*pi^2*D*t*(k/L).^2)));

ycalculated = @(x,t)(2*pi*D*t).^(-0.5).*exp(-(x.^2).*(4*D.*t).^(-1));
plot(x,y1);
hold on;
plot(x,y2);
plot(x,y3);
plot(x,y2(nk+1)/ycalculated(0,0.1)*ycalculated(x,.001));
plot(x,y2(nk+1)/ycalculated(0,0.1)*ycalculated(x,.1));
plot(x,y2(nk+1)/ycalculated(0,0.1)*ycalculated(x,.2));
title("Heat Equation Solution Time Slices")
xlabel("Space x")
ylabel("Temperature y")
legend("FFT t=0.001", "FFT t=0.1", "FFT t=0.2", "Dirac t=0.001", "Dirac t=0.1", "Dirac t=0.2")
hold off

```



TASK 1

The heat equation describes how the temperature spreads out over time. Calculate $\text{sum}(y1)$, $\text{sum}(y2)$ and $\text{sum}(y3)$. What do those terms represent? Why are they the same?