Lecture 6

Laplace Transformation

$$F(s) = L\{f(t)\} = \int_0^\infty e^{-st} f(t) dt$$

Laplace Transforms convert derivatives in t-space into algebraic equations in s-space. For example

$$L\left\{\frac{d}{dt} f(t)\right\} = \int_0^\infty e^{-st} \frac{d}{dt} f(t) dt$$

applying chain rule

$$= \int_0^\infty \frac{d}{dt} [e^{-st} f(t)] - f(t) \frac{d}{dt} e^{-st} dt$$

$$= e^{-s(\infty)} f(\infty) - e^{-s(0)} f(0) + \int_0^\infty s e^{-st} f(t) dt$$

assuming the function f is finite for t -> infinity

$$= 0 \cdot f(\infty) - 1 f(0) + s \int_0^\infty e^{-st} f(t) dt$$

$$L\{f'(t)\} = s \cdot L\{f(t)\} - f(0)$$

We can verify this using Matlabs symbolic calculation engine

```
syms t f(t)
laplace(diff(f(t),t))
```

ans = $s \, \mathrm{laplace}(f(t), t, s) - f(0)$

Similarly, taking the second derivative as the derivative of the first derivative and applying the rule above

$$L\left\{\frac{d^2}{dt^2} f(t)\right\} = L\left\{\frac{d}{dt}\left(\frac{d}{dt} f(t)\right)\right\}$$

$$= s \cdot L\left\{\frac{d}{dt} f(t)\right\} - f'(0)$$

$$= s^2 \cdot L\{f(t)\} - s \cdot f(0) - f'(0)$$

$$L\{f''(t)\} = s^2 \cdot L\{f(t)\} - s \cdot f(0) - f'(0)$$

```
syms t f(t)
laplace(diff(diff(f(t),t),t))
```

ans =

$$s^2 \, \text{laplace}(f(t), t, s) - s \, f(0) - \left(\left(\frac{\partial}{\partial t} f(t) \right) \bigg|_{t=0} \right)$$

Let's also consider applying the Laplace transform to an integral $F(t) = \int_0^t f(\tau)d\tau$ with "initial condition"

$$F(0) = \int_0^0 f(\tau)d\tau = 0$$

$$L\{F(T)\} = \int_0^\infty e^{-st}F(t)dt$$

Again, it's helpful to apply integration-by-parts. Taking the derivative of the integrant

$$\frac{d}{dt}\left[e^{-st}F(t)\right] = -se^{-st}F(t) + e^{-st}f(t)$$

Integrating both sides and dividing

$$\left[e^{-st}F(t)\right](\infty) - \left[e^{-st}F(t)\right](0) = -s\int_0^\infty e^{-st}F(t)dt + \int_0^\infty e^{-st}f(t)dt$$

The right hand side vanishes at both limits, towards infinity the exponential suppresses the growth of F(t) while at the lower limit the integral F(0) is by design zero.

$$0 = -s\int_0^\infty e^{-st}F(t)dt + L[f(t)]$$

In summary

$$L\{F(t)\} = \frac{L\{f(t)\}}{s}$$

To be able to verify the Laplace transform above with Matlab we need be careful to use a function variable f(d) that is different from the upper bound "t" of the symbolic integral function "int":

```
syms t d f(d)
laplace(int(f,0,t))
```

```
ans =
```
$$\frac{\text{laplace}(f(t), t, s)}{s}$$

In summary the Laplace transform converts derivatives in t-space into algebraic equations of the transform and its initial conditions in s- space

$$L\left\{ \int_0^t f(\tau)d\tau \right\} = \frac{L\{f(t)\}}{s}$$

$$L\{f'(t)\} = s \cdot L\{f(t)\} - f(0)$$

$$L\{f''(t)\} = s^2 \cdot L\{f(t)\} - s \cdot f(0) - f'(0)$$

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Laplace Transforms of Polynomial functions

Powers of t, starting with $t^0 = 1$

$$L\{1\} = \int_0^\infty e^{-st}\mathrm{dt} = \left[\frac{1}{-s}e^{-st}\right]_0^\infty = 0 - \left(-\frac{1}{s}\right) = \frac{1}{s}$$

Next look at the general case for t^n. Start with chain rule to generate an 'integration by parts' pattern

$$\frac{d}{\mathrm{dt}}\left[t^n e^{-st}\right] = nt^{n-1}e^{-st} - st^n e^{-st}$$

$$t^n e^{-st} = \frac{n}{s}t^{n-1}e^{-st} - \frac{1}{s}\frac{d}{\mathrm{dt}}\left[t^n e^{-st}\right]$$

Integrating

$$\int_0^\infty t^n e^{-st}\mathrm{dt} = \frac{n}{s}\int_0^\infty t^{n-1}e^{-st}\mathrm{dt} - \frac{1}{s}\left[\infty^n e^{-\infty} - 0^n e^0\right]$$

Thus we get the following recursive rule for Laplace transforms for powers of t

$$L\{t^n\} = \frac{n}{s}L\{t^{n-1}\}$$

which builds up succesively from t^0

$$L\{t^n\} = \frac{n}{s}L\{t^{n-1}\} = \frac{n}{s}\frac{n-1}{s}L\{t^{n-2}\}$$

$$= \frac{n!}{s^n}L\{t^{n-n}\} = \frac{n!}{s^n}L\{1\} = \frac{n!}{s^{n+1}}$$

Another importent case is to consider to an exponential shift on the Laplace tranform

$$L\{e^{at}f(t)\} = \int_0^\infty e^{at}f(t)e^{-st}\mathrm{dt} = \int_0^\infty f(t)e^{-(s-a)t}\mathrm{dt}$$

Thus

$$L\{e^{at}f(t)\} = F(s-a) \text{ , with}$$

$$L\{f(t)\} = F(s)$$

In summary we can make a table for increasing powers of t

$f(t)$	$F(s)$	$e^{at}f(t)$	$F(s-a)$
$\delta(t)$	1	$e^{at}\delta(t)$	1
1	$\frac{1}{s}$	e^{at}	$\frac{1}{s-a}$

t | $\dfrac{1}{s^2}$ | te^{at} | $\dfrac{1}{(s-a)^2}$

t^2 | $\dfrac{2}{s^3}$ | $t^2 e^{\text{at}}$ | $\dfrac{2}{(s-a)^3}$

```
syms t f(t) n
assume(n,"integer")
laplace(t)
```

ans =

$\dfrac{1}{s^2}$

```
laplace(t^2)
```

ans =

$\dfrac{2}{s^3}$

```
laplace(t^n)
```

ans =

$$\begin{cases} \dfrac{\Gamma(n+1)}{s^{n+1}} & \text{if } 0 \le n \end{cases}$$

Next consider the Laplace transform of a cos

Expressing the cos in terms of complex exponentials so that we can apply the formulas from above

$$\cos(\omega t) = \frac{1}{2} e^{j\omega t} + \frac{1}{2} e^{-j\omega t}$$

$$L\{\cos(\omega t)\} = \frac{1}{2} L\{e^{j\omega t}\} + \frac{1}{2} L\{e^{-j\omega t}\}$$

$$= \frac{1}{2} \frac{1}{s - j\omega} + \frac{1}{2} \frac{1}{s + j\omega} = \frac{1}{2} \frac{s + j\omega + s - j\omega}{s^2 + \omega^2}$$

$$L\{\cos(\omega t)\} = \frac{s}{s^2 + \omega^2}$$

```
syms t w
laplace(cos(w*t))
```

ans =

$\dfrac{s}{s^2 + w^2}$

```
laplace(sin(w*t))
```

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ans =

$$\frac{w}{s^2 + w^2}$$

Laplace Transforms for source functions: Impulse, Step and Ramp

Delta function, a single impulse at t=0. The delta function represent an unit impulse of time width ϵ in the limit $\epsilon \to 0$ and

heigth (intensity) $\frac{1}{\epsilon}$ such that its integral of width *height of the impulse is always one. It's Laplace transform is

$$L\{\delta\,(t)\} = \int_0^\infty \delta\,(t)e^{-st}dt = \lim_{\epsilon \to 0}\int_0^\epsilon \frac{1}{\epsilon}e^{-st}dt = \lim_{\epsilon \to 0}\frac{1}{-s\epsilon}\left[e^{-s\epsilon} - 1\right]$$

expanding the exponential for small arguments with $e^x = 1 + x + \frac{1}{2}x^2 + \dots$

$$\lim_{\epsilon \to 0}\frac{1}{-s\epsilon}\left[-s\epsilon + \frac{1}{2}(s\epsilon)^2 + \dots \text{higher powers of } (s\epsilon)\right] = 1 + \lim_{\epsilon \to 0}(\text{powers of } s\epsilon)$$

$$L\{\delta\,(t)\} = 1$$

The Heaviside Step function H(t-a)

$H(t-a) = 1$ for $t > a$
$H(t-a) = 0$ for $t < a$

Relation between the delta and the step function

$$H(t-a) = \int_{-\infty}^t \delta\,(\tau - a)d\tau$$

$$\delta\,(t-a) = \frac{d}{dt}H(t-a)$$

Laplace transform of the Heaviside function

$$L\{H(t-a)\} = \int_0^\infty H(t-a)e^{-st}dt = \int_a^\infty e^{-st}dt$$

$$L\{H(t-a)\} = \frac{e^{-sa}}{s}$$

Ramp function: t*H(t-a)

$$L\{t\,H(t-a)\} = \int_a^\infty te^{-st}dt = L\{t\} - \int_0^a te^{-st}dt$$

The first term is the Laplace transform of t

$$L\{t\} = \frac{1}{s^2}$$

Applying the chain rule to evaluate the second term

$$\frac{d}{dt}\left[te^{-st}\right] = e^{-st} - tse^{-st} \text{ or } te^{-st} = \frac{1}{s}e^{-st} - \frac{1}{s}\frac{d}{dt}\left[te^{-st}\right]$$

so that

$$\int_0^a te^{-st}dt = \frac{1}{s}\int_0^a e^{-st}dt - \frac{a}{s}e^{-st}\bigg|_{}^{} = -\frac{1}{s^2}e^{-sa} + \frac{1}{s^2} - \frac{a}{s}e^{-sa}$$

We get in total

$$L\{tH(t-a)\} = \frac{1}{s^2} - \left(-\frac{1}{s^2}e^{-sa} + \frac{1}{s^2} - \frac{a}{s}e^{-sa}\right)$$

$$L\{tH(t-a)\} = \frac{1}{s^2}e^{-sa} + \frac{a}{s}e^{-sa}$$

```
syms t a w f(t)
assume(a,"positive")
laplace(dirac(t-a))
```

```
ans = e^{-a s}
```

```
laplace(heaviside(t-a))
```

```
ans =

e^{-a s}
─────
  s
```

```
laplace(t*heaviside(t-a))
```

```
ans =

e^{-a s}   a e^{-a s}
────── + ────────
  s^2        s
```

Circuits in the Laplace domain

Resistor

$$V(t) = R \cdot I(t)$$

$$V(s) = R \cdot I(s) \text{ or } I(s) = \frac{V(s)}{R}$$

Capacitor

$$Q(t) = C \cdot V(t)$$

$$I(t) = C \cdot \frac{d}{dt} V(t)$$

Applying the Laplace transform to both sides $L\{I(t)\} = C \cdot L\{\frac{d}{dt} V(t)\} = C \cdot s \cdot L\{V(t)\} - C \cdot V(0)$

$$I(s) = C \cdot s \cdot V(s) - C \cdot V(0)$$

or

$$V(s) = \frac{I(s)}{s\,C} + \frac{V(0)}{s}$$

Inductor

$$V(t) = L \frac{d}{dt} I(t) \quad \text{(carfull, here L stands for the Inductance)}$$

Applying the Laplace transform to both sides

$$L\{V(t)\} = L\ L\{\frac{d}{dt} I(t)\}$$

$$V(s) = L\,s \cdot I(s) - L\,I(0)$$

or in terms of currents

$$I(s) = \frac{V(s)}{sL} - \frac{I(0)}{s}$$

Simple ODE (from week 5)

Solve

$$\frac{d}{dt} Y(t) = -0.1\,Y(t)$$

with initial condition Y(0)=25.

Apply Laplace transform on both sides.

$$L\{\frac{d}{dt} Y(t)\} = -0.1 L\{Y(t)\}$$

Use rule for derivatives under the Laplace transform

$$sL\{Y(t)\} - Y_0 = -0.1 L\{Y(t)\}$$

Solve in terms of the Laplace transform

$$L\{Y(t)\} = \frac{1}{s + 0.1} Y_0$$

Now we need to inverse transform back from s into t space

$$Y(t) = L^{-1}\left\{\frac{1}{s+0.1}\right\}Y_0$$

From the table of Laplace transformations of powers of t that we use

$$L\{e^{at}\} = \frac{1}{t-a}$$

$$e^{at} = L^{-1}\left\{\frac{1}{t-a}\right\}$$

so that we get for $a = -1$ and $Y_0 = 25$

$$Y(t) = e^{-0.1t}Y_0 = 25\, e^{-0.1t}$$

which agrees with what we got in the previous lecture using either direct integration or Matlab's symbolic solver

```
syms Y(t) t
dsolve(diff(Y(t),t)==-0.1*Y(t),Y(0)==25)
```
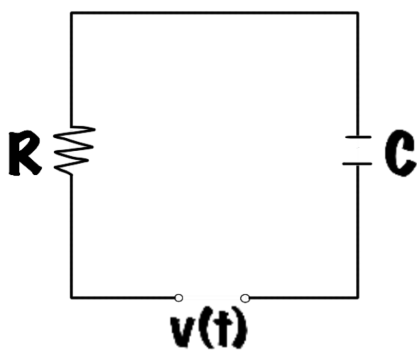
```
ans =
```

$$25\, e^{-\frac{t}{10}}$$

RC Circuit

Consider a simple circuit that consists of a resistor R and capacitor C in series attached to a voltage source V. Initially the voltage source is zero and the capacitor is uncharged. At time t=0 the voltage linearly increases until it reaches 50V at t=5sec. Afterwards it stays at that level. Use R = 5 Ohm and C = 0.2 F

Task: Compute the time dependent current I(t) using the Laplace transform methodology. In this example we make use of Matlabs symbolic functionality to solve the algebra and generate the Laplace transformations

The voltages at the resistor and the capacitor add to the source voltage

$$V(t) = V_R(t) + V_c(t) = R\, I + \frac{Q(t)}{C} = R\, I(t) + \frac{1}{C}\int_0^t I(\tau)d\tau$$
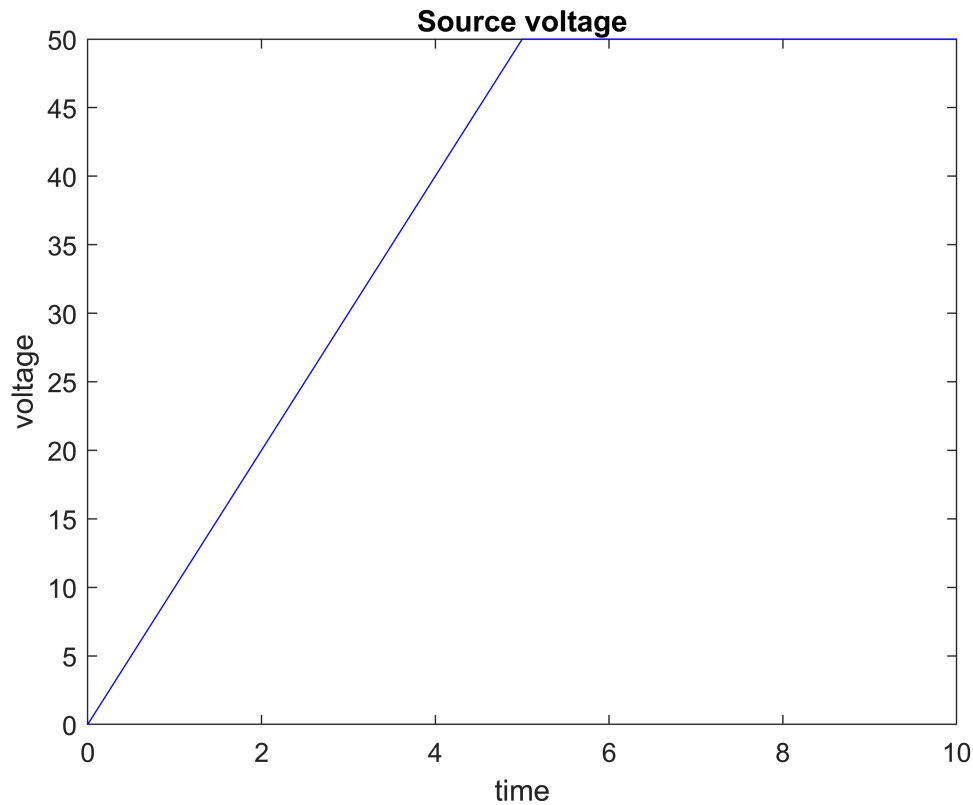
The source voltage can be composed out of a combination of step functions

$$V(t) = 10\,t\,H(t) - 10\,t\,H(t-5) + 50\,H(t-5)$$

```
syms t
V(t)= 10*t*heaviside(t)-10*t*heaviside(t-5)+50*heaviside(t-5)
```

V(t) = $50\,\mathrm{heaviside}(t-5) - 10\,t\,\mathrm{heaviside}(t-5) + 10\,t\,\mathrm{heaviside}(t)$

```
plot(0:.1:10,V(0:.1:10),"b-")
title("Source voltage")
xlabel("time")
ylabel("voltage")
```


The Laplace transform of the source voltage terms is

```
SRC = laplace(V(t),t,s)
```

SRC =

$$\frac{10}{s^2} - \frac{10\,\mathrm{e}^{-5\,s}}{s^2}$$

The Laplace transfrom of the circuit current terms is

```
syms I(t) C R d X
LCRT = laplace(R*I(t)+1/C*int(I(d),0,t))
```

LCRT =

$$R \, \text{laplace}(\text{I}(t), t, s) + \frac{\text{laplace}(\text{I}(t), t, s)}{C \, s}$$

Let's assign the variable X to represent the Laplace transform of I(t)

```
CRT=subs(LCRT,laplace(I(t),t,s),X)
```

CRT =

$$R \, X + \frac{X}{C \, s}$$

Defining the circuit equation by equaling the left and right hand sides

```
eq=SRC==CRT
```

eq =

$$\frac{10}{s^2} - \frac{10 \, e^{-5 \, s}}{s^2} = R \, X + \frac{X}{C \, s}$$

Solving for the Laplace transform X

```
eqX = solve(eq,X)
```

eqX =

$$-\frac{10 \, C - 10 \, C \, e^{5 \, s}}{s \, e^{5 \, s} + C \, R \, s^2 \, e^{5 \, s}}$$

Substituting values for the capacitance and resistance

```
eqXS = subs(eqX,[C R],[.2 5])
```

eqXS =

$$\frac{2 \, e^{5 \, s} - 2}{s \, e^{5 \, s} + s^2 \, e^{5 \, s}}$$

Inversting the Laplace transform   (somehow Matlab does not do it)
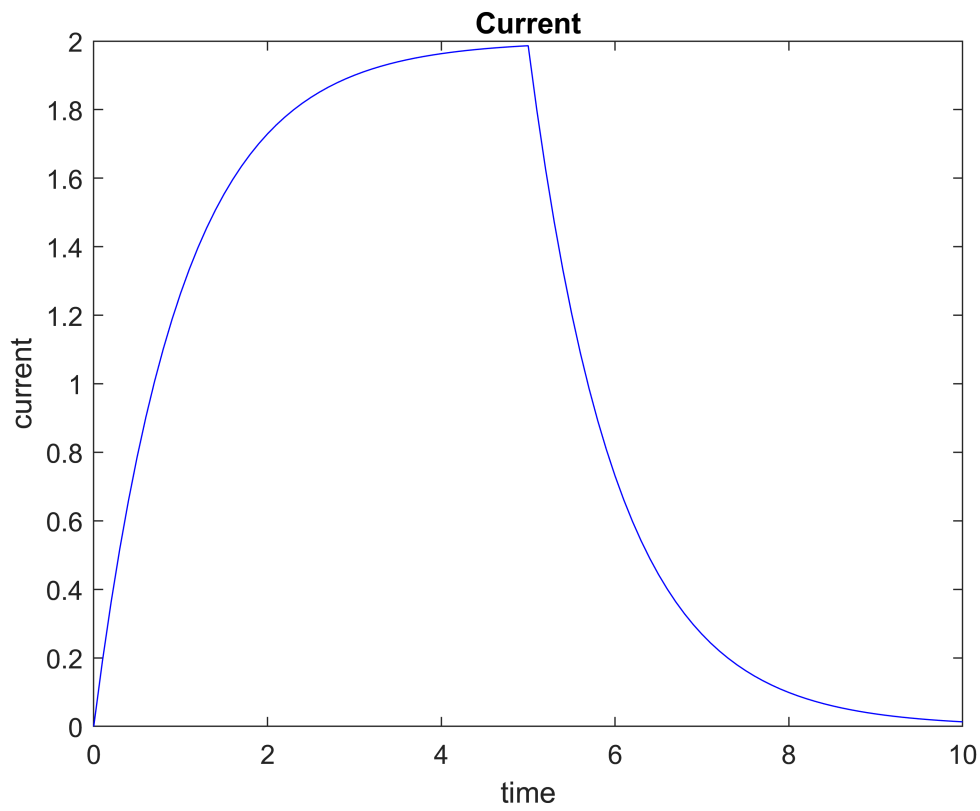
```
ilaplace(eqXS)
```

ans =

$$2 \, \text{ilaplace}\left(\frac{e^{5 \, s}}{s \, e^{5 \, s} + s^2 \, e^{5 \, s}}, s, t\right) - 2 \, \text{ilaplace}\left(\frac{1}{s \, e^{5 \, s} + s^2 \, e^{5 \, s}}, s, t\right)$$

However the inverse works if one retypes the expression. The inverse Laplace transform gives the current.

```
I(t) = ilaplace(2*(1-exp(-5*s))/(s*(s+1)))
```

I(t) = $2 \, \text{heaviside}(t - 5) \, (e^{5-t} - 1) - 2 \, e^{-t} + 2$

```
plot(0:.1:10,I(0:.1:10),'b-')
title("Current")
xlabel("time")
ylabel("current")
```

Current

TASK:

Repeating similar steps as above, find the current I(t) in the LC circuit below, with L = 1H, C = 1F and the source voltage is V(t) = 1-exp(-t) for 0<t<PI and V(t)=0 for t>PI. Assume zero initial current and charges.