Lecture 5

Solving ODEs with Matlab

Matlab provides functionalities to support solving differential equation. It can generate a numerical simulation for an ordinary differencial equation given a concrete starting point and a desired range of values for the independent variable (time axis). In some cases Matlab's symbolic algebra engine can also be used to derive a closed form solution.

Consider for example the following simple first order ODE

$$\frac{\mathrm{dy}}{\mathrm{dt}} = -0.1\, y$$

This can be solved in closed form by integration

$$\int \frac{1}{y}\mathrm{dy} = -0.1 \int \mathrm{dt}$$

$$\ln(y) - \ln(y_0) = -0.1(t - t_0)$$

$$y = y_0 e^{-0.1(t-t_0)}$$

To be able to utilize Matlab's symbolic solver we first register **y(t)** and its derivative as diff(y,t) as symbolic variables and define the differential equation (ode)

```
syms y(t)
ode = diff(y,t) == -0.1*y
```

ode(t) =

$$\frac{\partial}{\partial t}\, y(t) = -\frac{y(t)}{10}$$

Optionally we can also set an initial condition

```
cond = y(0) == 25
```

cond = $y(0) = 25$

Finally, using **dsolve** generates the solution

```
ySol(t) = dsolve(ode,cond)
```

ySol(t) =

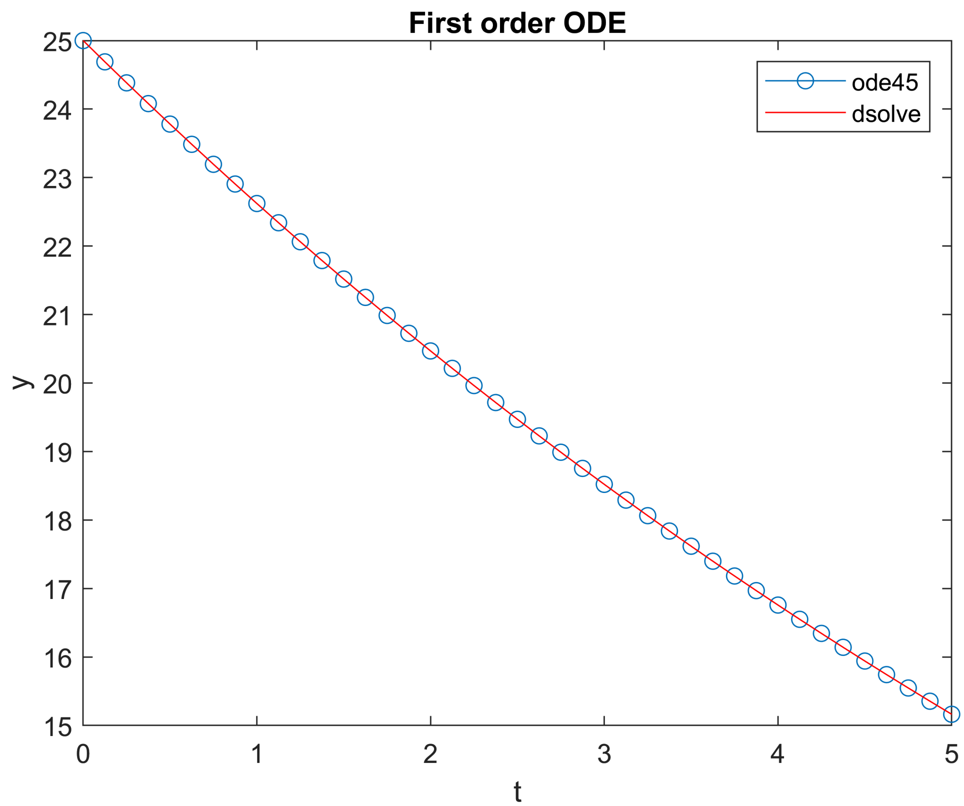
$$25\, \mathrm{e}^{-\frac{t}{10}}$$

It is not always possible to derive a closed form symbolic solution. To generate a numerical solution, we first provide a desired range for the independent t variable as well as the starting value for the solution y

```
tspan = [0 5];
y0 = 25;
```

Finally we generate the solution using the Matlab function **ode45** by supplying the first derivative as an anonymous @ function. Plotting both, the symbolic and numerically generated solution shows an agreement

```
[t, y] = ode45( @(t,y) -0.1*y, tspan, y0 );
plot(t,y,'-o');
hold on;
plot(t,ySol(t),'r-');
title("First order ODE");
legend("ode45","dsolve");
xlabel("t");
ylabel("y");
hold off;
```



On the other hand Matlab is not able to derive a closed form solution for the following equation, which is non-linear and inhomogenous

```
syms y(t)
ode = diff(y,t) == 1-y^4
```

ode(t) =

$$\frac{\partial}{\partial t}\, y(t) = 1 - y(t)^4$$

```
ySol(t) = dsolve(ode,cond)
```

Warning: Unable to find explicit solution. Returning implicit solution instead.

ySol(t) = $\mathrm{solve}(-\mathrm{atan}(y) - \mathrm{atanh}(y) = -2\,t - \mathrm{atan}(25) - \mathrm{atanh}(25),\, y)$

The equation above is of the form of simple climate model, describing the change in earth temperature T that results from the balance between sunlight energy inflow Ein and reflection and 'black body' radiation outflow Eout, see

https://www.e-education.psu.edu/meteo469/node/137

$$\frac{\mathrm{dT}}{\mathrm{dt}} = E_{\mathrm{in}} - \epsilon\, E_{\mathrm{out}}$$

with

$$E_{\mathrm{in}} = \frac{Q}{R}(1 - \alpha)$$

$$E_{\mathrm{out}} = \frac{\sigma}{R} T^4$$

Common parameter values are

R = 2.912 W-yr/m2K earth surface heat capacity

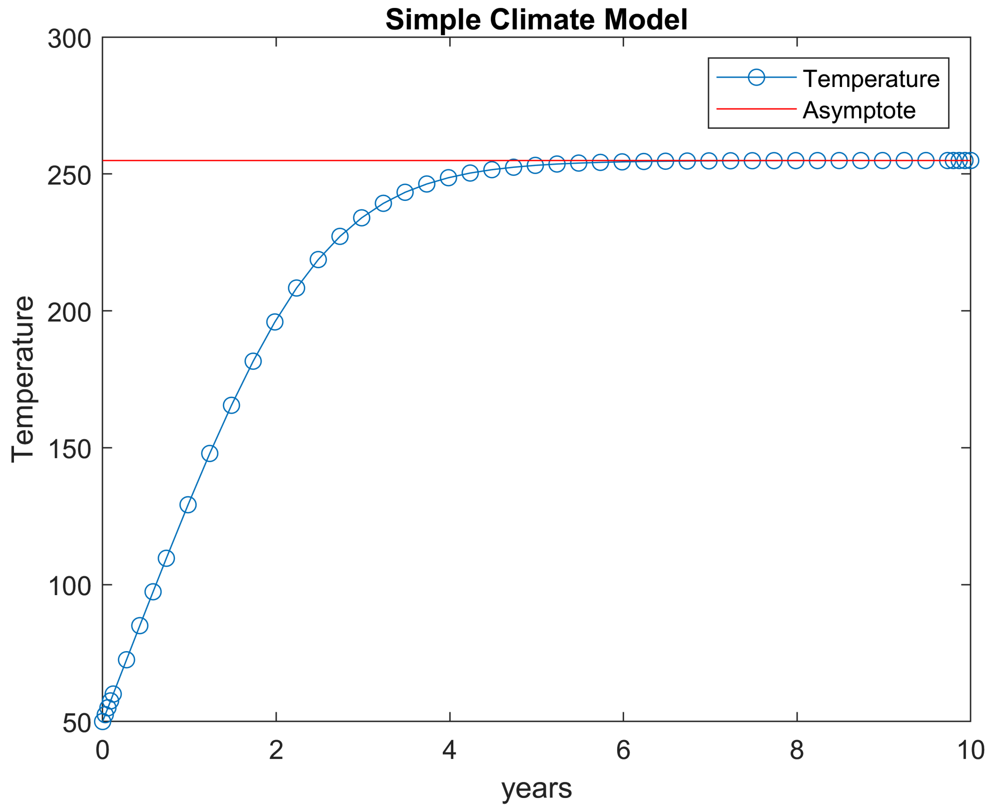
Q = 342 W/m2 Mean incoming sunlight radiation

alpha = 0.30   Albeto = light reflected by surface

sigma = 5.67E-8 W/m2 K4

epsilon = 1   parameter that mediates energy in and outflow

```
R=2.912;
Q=342;
alpha = 0.3;
sigma = 5.67E-8;

tspan = [0 10];
T0 = 50;
[t, T] = ode45( @(t,T) ( Q/R*(1 - alpha) - sigma/R*(T)^4),tspan, T0);
plot(t,T,'-o');
hold on;
plot(tspan,[(Q*(1-alpha)/sigma)^(1/4) (Q*(1-alpha)/sigma)^(1/4)],'r-');
title("Simple Climate Model");
xlabel("years");
ylabel("Temperature");
legend("Temperature", "Asymptote");
hold off;
```



We can find a fixed point, or asymptotic solution of the ODE by setting the right hand side of the ODE to zero

```
(Q*(1-alpha)/sigma)^(1/4)
```

```
ans = 254.9091
```

Alternative we can use Matlabs root finding function **fzero** to search for the value where the right-hand-side of the ODE is zero, provided we give a close enough starting value

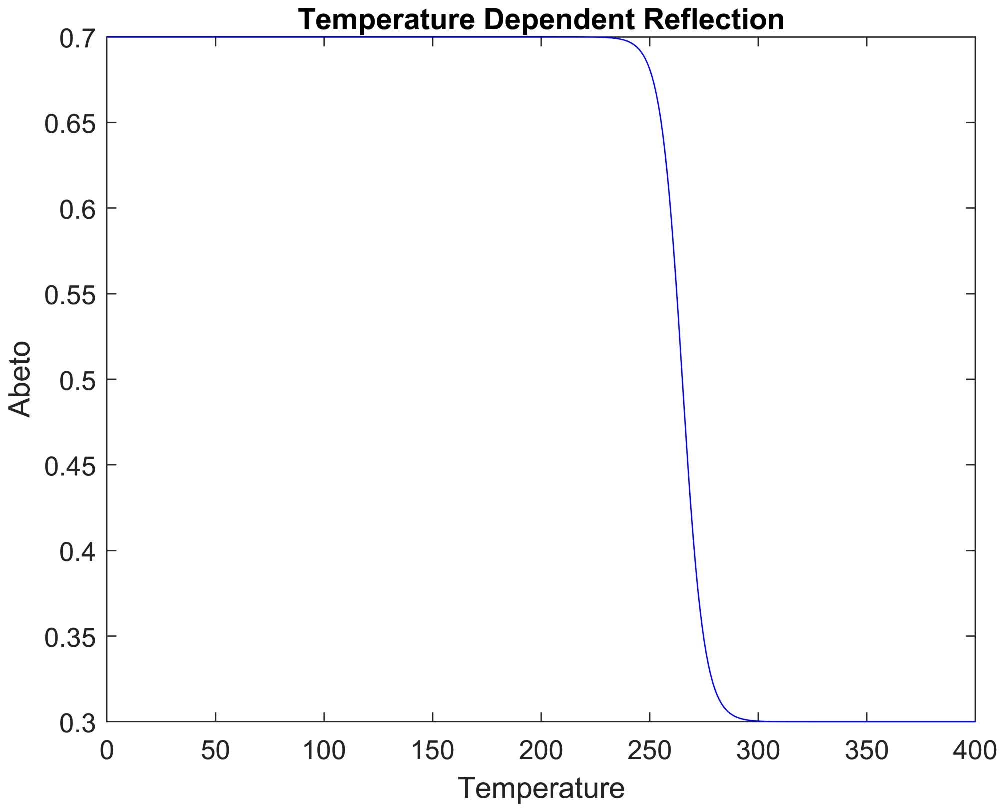
```
fzero( @(T) ( Q/R*(1 - alpha) - sigma/R*(T)^4), 200)
```

```
ans = 254.9091
```

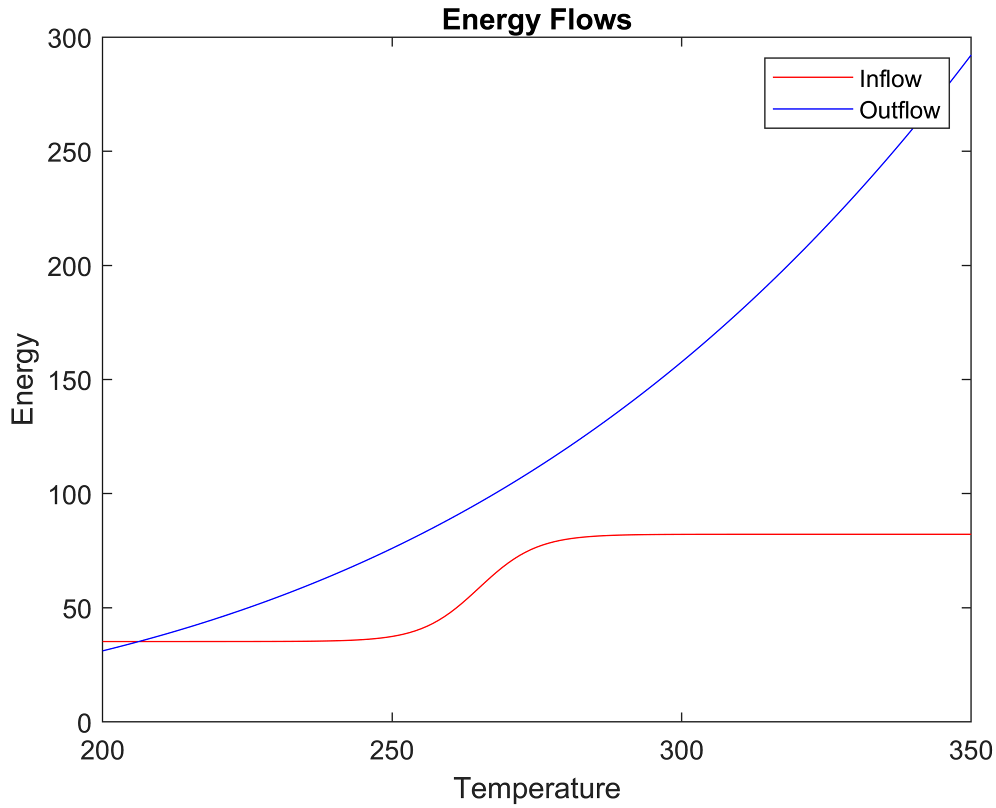
The model can be further refined by allowing the reflection value alpha to vary with temperature T between within a range using

αlpha(T)=0.5+0.2 tanh(0.1(265 − T))

```
alp= @(T) 0.5+0.2*tanh(0.1*(265 -T ));

plot([0:1:400],alp([0:1:400]),'b-');
title("Temperature Dependent Reflection");
xlabel("Temperature");ylabel("Abeto");
```

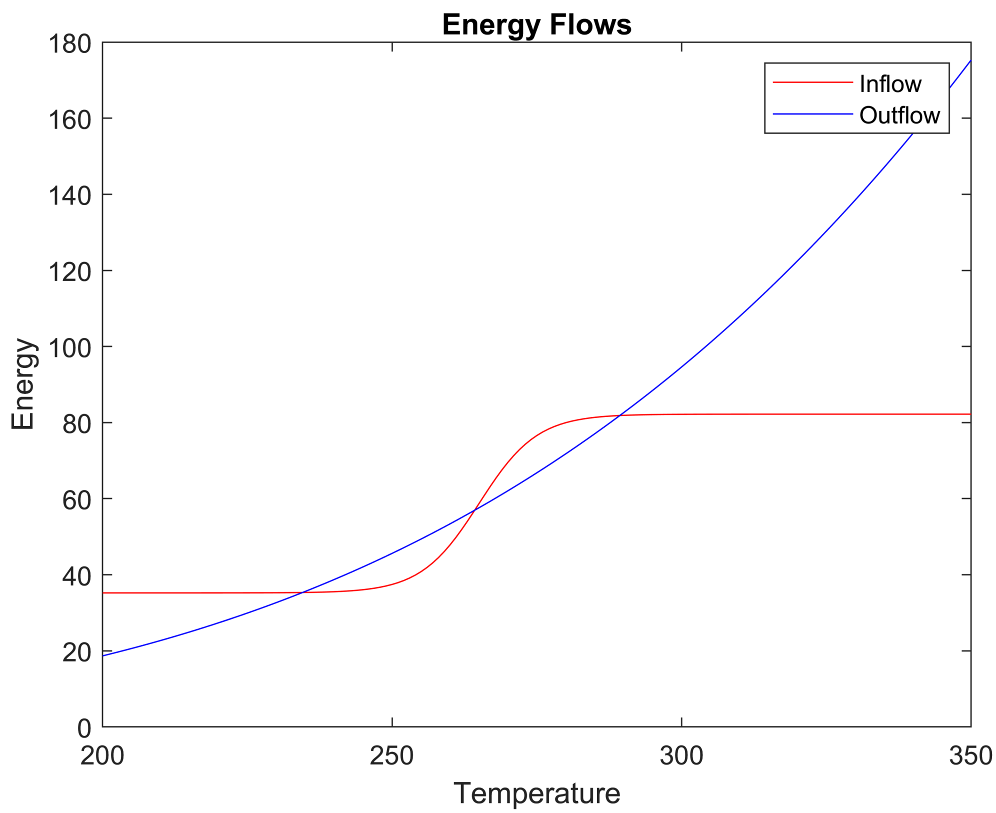



```
Ein = @(T) Q/R*(1-alp(T));
Eout =  @(T) sigma/R*(T).^4;

eps=1;
dTdt = @(t,T) Ein(T) - Eout(T);

plot([200:1:350],Ein(200:1:350),'r-')
hold on
plot([200:1:350],1.0*Eout(200:1:350),'b-')
title("Energy Flows");legend("Inflow", "Outflow");xlabel("Temperature");ylabel("Energy")
hold off
```



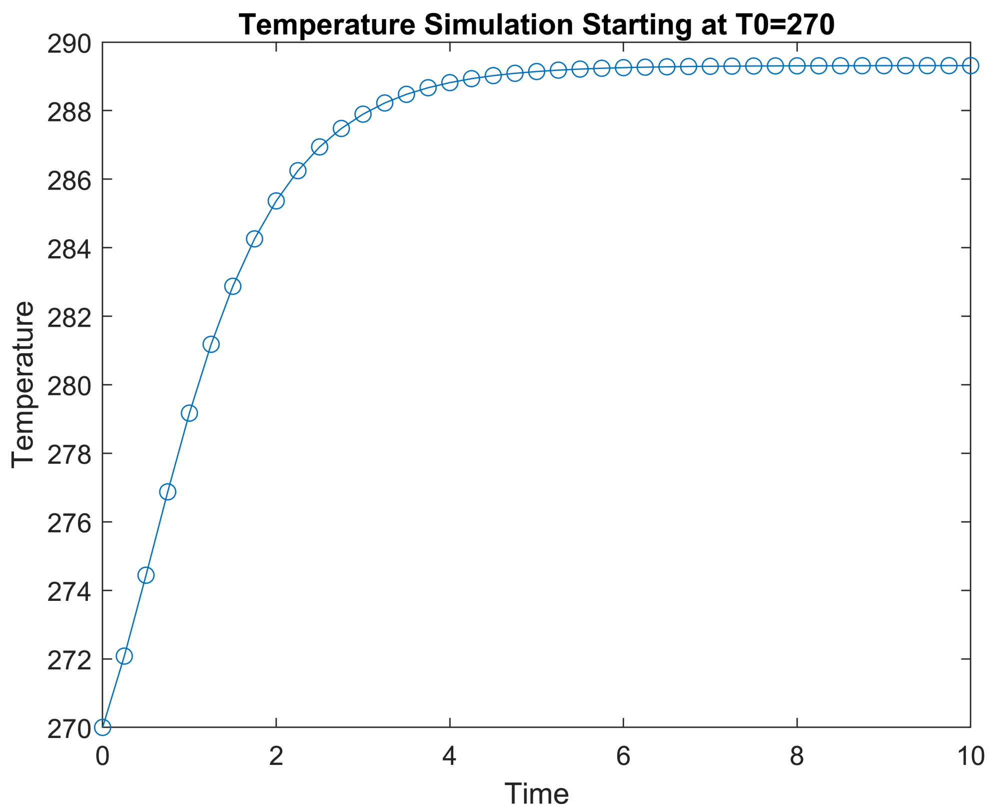
We can find the equilibrium temperature where the inflow and outflow curves meet. In the case above its at a low value to the left. The parameter epsilon tunes the outflow, perhaps as an result of changes in the atmosphere. Using epsilon as 0.6 we find

```
plot([200:1:350],Ein(200:1:350),'r-')
hold on
plot([200:1:350],0.6*Eout(200:1:350),'b-')
title("Energy Flows");legend("Inflow", "Outflow");xlabel("Temperature");ylabel("Energy")
hold off
```

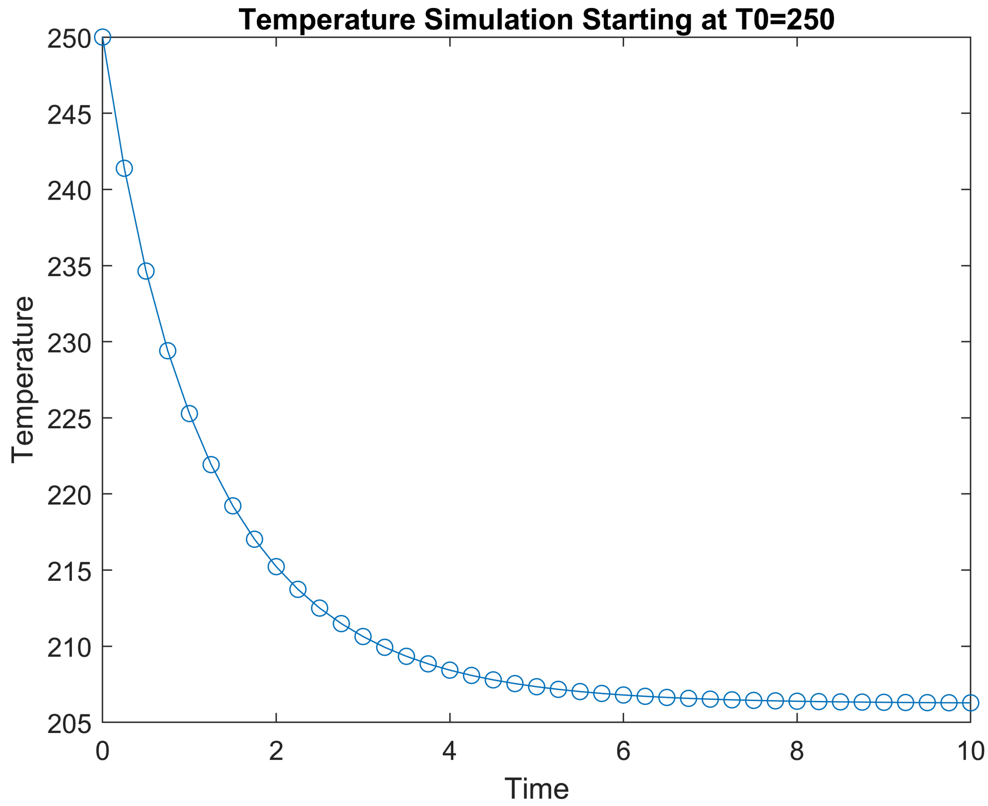


Now we have three possible fixed points where in and outflow balance! The asymptotic value where the tempurature eventually settles depends on the initial condition. For example starting with T0=270

```
T0=270;
[t, T] = ode45( @(t,T) Ein(T) - 0.6*Eout(T),tspan, T0);
plot(t,T,'-o');
title("Temperature Simulation Starting at T0=270");
ylabel("Temperature");
xlabel("Time");
```




```
T0=250;
[t, T] = ode45( @(t,T) Ein(T) - 1.0*Eout(T),tspan, T0);
plot(t,T,'-o');
title("Temperature Simulation Starting at T0=250");
ylabel("Temperature");
xlabel("Time");
```



Tasks:

Find the values of the temperatures of the three fixed points where the Ein and Eout energy flows balance for epsilon=0.6? Hint, use fzero() with appropriate starting points

Repeat the simulation above for different starting points T0. What values of T0 result in either the upper, lower of middle temperature?

Why do different staring points yield different asympotitic end temperatures?

Solve the following equations with starting values y(0)=10. Can you do it analytically with Matlab or even manually with pencil and paper? Generate a numerical solution.

$$\frac{\mathrm{dy}}{\mathrm{dt}} = 1 - y$$

$$\frac{\mathrm{dy}}{\mathrm{dt}} = -y^2$$

$$\frac{\mathrm{dy}}{\mathrm{dt}} = 1 - y^2$$