Logic Workshop

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CITYTECH OPENLAB:

EXPLORING QUANTITATIVE REASONING

- Basic Concepts:

Statements, Truth Value, Logical Connectives

- Constructing Truth Table
- Valid/Invalid argument

Logic

- Thinking critically
- Correct Reasoning
- Proof
- Evaluating argument – validity

In logic,

We use letters p, q, r … to represent statements.

a declarative sentence that has TRUTH VALUE

Either true or false
→ T / F

p: "It is raining" "Rocks exist" "Nets won the game yesterday"

q: "Today is Friday" "Bananas are pink" "2-1=4"

"That's awesome!" "Please close the door." "What time is it now?"

connectives

- A compound statement is a sentence formed by joining two or more simple statements with a connective.

Name	Connective	symbol
Conjunction	“and”	$\wedge$ (&)
Disjunction	“or”	$\vee$
Conditional	“if ..then”	$\rightarrow$
Biconditional	“if and only if”	$\leftrightarrow$
Negation	“not”	~

Practice

- $\wedge$
- $\rightarrow$
- $\vee$
- ~
- $\leftrightarrow$

p: "It is raining"
q: "Today is Friday"
r: "There will be no MAT 1190 class"

$p \wedge q$: "It is raining and today Friday"
$p \vee q$:
$\sim p \rightarrow q$
$p \leftrightarrow q$
$\sim p$
$q \rightarrow r$

Quantifiers

- **Existential**

Some

Sometimes

There is

There exist

- **Universal**

All

Always

None

Never

Truth Table

- Truth tables are visual aids to help us determine all the truth value possibilities of various statements.
- We construct truth table to identify various distinctions (such as tautologies, self-contradictions, consistent statements, equivalent statements, and valid arguments).

How to Read Truth Table

These are statements
The first row says "p" is true, "q" is true, and "p ^q" is true.
The second row says "p" is true, "q" is false, and "p^q" is false.
The third row says "p" is false, "q" is true, and "p^q" is false.
The fourth row says "p" is false, "q" is false, and "p^q" is false.

p	q	p ∧ q
T	T	T
T	F	F
F	T	F
F	F	F

- There is a column (vertical area) under each statement, which contains every possible truth value. The column under "p" has "T, T, F, F". The column under "q" is "T, F, T, F" . The column under "p ∧ q" contains "T, F, F, F" .
- Every row (horizontal area) beneath the statements contains every combination of truth values.

Truth Tables for the Connectives & Negation, Tautology, Self-contradiction

p	q	p ∧ q
T	T	T
T	F	F
F	T	F
F	F	F

P	q	p V q
T	T	T
T	F	T
F	T	T
F	F	F

p	q	p → q
T	T	T
T	F	F
F	T	T
F	F	T

P	q	p ↔ q
T	T	T
T	F	F
F	T	F
F	F	T

p	~p
T	F
F	T

p	~p	p V ~p
T	F	T
F	T	T

P	~P	P ∧ ~P
T	F	F
F	T	F

Negation

p	~p
T	F
F	T

- “p” is any possible statement and “~p” means “it’s not the case that p.”

Tautologies

P	~P	P V ~P
T	F	T
F	T	T

A tautology is a statement that is always true because of it's logical form.

Ex. "there are life forms on other planets or there are no life forms on other planets."

This statement has the form "P V ~P."

The truth table above shows that all the possible truth values of "P V ~P" are true.

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cf) Self-contradictions

A self-contradiction is a statement that's always false.

For example, "there are life forms on other planets and there are no life forms on other planets." That statement has the form "p ∧ ~p."

P	~P	P ∧ ~P
T	F	F
F	T	F

Biconditional

Biconditional

: p if and only if q

P	q	p $\leftrightarrow$ q
T	T	T
T	F	F
F	T	F
F	F	T

The table above makes it clear that "p $\leftrightarrow$ q" is only true when "p" and "q" have the same truth values. They must both be true or false. If not, the statement is false.

Conjunction

- Conjunction: p and q

P	q	P ∧ q
T	T	T
T	F	F
F	T	F
F	F	F

- The truth table makes it clear that "p ∧ q" is true only when all the components (in this case, "p" and "q") are true.
- Every other case is false. The ***whole*** statement "p ∧ q" must be false if any one of the components is false.

- Ex. "I got an p in logic class and I got an p in MAT1190" →what would be the truth value of the whole statement if you didn't get an a in logic while p in the MAT1190?
- p: "humans are animal" q: "humans are mammal."
- P: "Paris is in France" q: "Berlin is in German"

Disjunction

- Disjunction: either p or q

P	q	p V q
T	T	T
T	F	T
F	T	T
F	F	F

- The truth table indicates that every “p V q” statement is true **unless** both “p” and “q” are false, which is shown on the final row down.
- Disjunctive statement is false only when all of the components (in this case, “p” and “q”) are false.

Ex. Suppose you make a statement “I will buy a new iPad or a new smart phone.”
If you buy neither, then the whole disjunctive statement would be false. If you actually did buy one or the other, then this statement would be true. The statement will be considered true if you buy both of them.

Conditional

$p \rightarrow q$

"p→q" means "if p then q" ("p" is an antecedent "q" is a consequence.); *as long as the antecedent p is the case*, the consequence q will be the case.

p	q	p → q
T	T	T
T	***F***	***F***
F	T	T
F	F	T

"p → q" is true <u>unless</u> "p" is true and "q" is false; when p is T and q is F then p→q is F, and every other case is T.

Making Sense of Conditional with Examples

EXAMPLE

- Prof. Kang said, “If you submit a summary paper, then I will give you an A.”
- Four Possible Situations:
  - ❑ You did submit the paper, so you get an A. (p:T q:T)
  - ❑ You did submit it, but she didn’t give you an A. (p:T q:F)
    - ➔You’d say prof. K LIED! (her statement is FALSE!)
  - ❑ *You did NOT submit it*, and you get an A. (p:F q:T)
    - ➔You will NOT say her statement is FALSE, because you didn’t submit it in the first place!
  - ❑ You did NOT submit it, and you didn’t get an A. (p:F q:F)
    - ➔You will NOT say her statement is FALSE, because you didn’t submit it in the first place!

Example 1.

Let’s imagine that a politician claims the following compound statement:

“If I win the election, then taxes will go down.”

p: “I win the elections”

q: “Taxes will go down”

p	q	p → q
T	T	T
T	***F***	***F***
F	**T**	**T**
F	**F**	**T**

- Row 1 (possibility 1)

Let's say he or she (the politician) won, and taxes went down. (p: T, q: T) Then the whole statement "p->q" is True. S/he didn't lie.

- Row 2 (possibility 2)

Let's say, s/he won (p: T), but taxes haven't gone down (q: F). Then the whole statement "p->q" is False. S/he lied!

BUT what if he didn't win, in the first place (p, the antecedent is false)?

p → q becomes false
only when s/he wins
and (yet) the taxes haven't gone down!

p	q	p → q
T	T	T
T	F	F
F	**T**	**T**
F	**F**	**T**

- Row 3 (possibility 3)

Let's say s/he didn't win (p: F), and still taxes went down (q: T).

Good for taxpayers! Yet, we should note that the consequence q happens to be true regardless of q's being the case.

We have no clue to say that s/he lied; what s/he claimed is based on the assumption that "s/he won the election." Given that the antecedent p is false, the statement (p->q) is not false.

- Row 4 (possibility 4)

How about s/he didn't win (p: F) and taxes haven't gone down (q: F)? Again, we cannot say s/he lied. What is declared in "p->q" is "If s/he won" the stated consequence will occur. If s/he didn't win in the first place, then we are not allowed to accuse her/him not to make q happen.

Example 2

p: The Nets win tomorrow q: They make the playoffs

p	q	p → q
T	T	T

- Possibility 1:

The Nets win (p:T), and they make the playoffs(q:T).

p	q	p → q
T	***F***	***F***

- Possibility 2

Given that one claims that “If the Nets win tomorrow, they make the playoffs,” the consequence (q: making playoffs) must follow from the antecedent (p: win)!

In the case in which the consequence does not occur (q:F) upon the antecedent (p:T), p→q must be false.

Example 2

p	q	$p \rightarrow q$
F	T	T

- Possibility 3

How about the possibility in which the Nets lose tomorrow and still make the playoffs (p: F q:T)?

Recall what is claimed: *if the Nets won*, they would make the playoffs.

In order for that claim (the whole statement) to be false, the Nets would have to win and not make the playoffs (as possibility 2). However, that is not the case if they didn't win, so the statement is not false, i.e., true!

- Possibility 4

p	q	p → q
F	F	T

In this possible case, the Nets lose tomorrow (p:F) and don’t make the playoffs (q:F).

Let’s again recall the initial claim p->q is based on the following condition: (if) Nets win. → The compound statement is <u>false only if</u> the Nets win and don’t make the playoffs!

If the Nets lose (if p is false), the statement is not false, making it true.

Key idea!
Truth Values
for a Conditional Statement

- Note that what is claimed by a compound statement p → q is:

  Given the antecedent is the case, the consequence will be the case.

  This in turn means, when the antecedent p is not true in the first place, then we cannot say the whole statement p → q is false, making it true!
- p → q becomes false *only when p* is true yet q is not true!

Various Ways of Stating/Writing a Conditional Statement

p→q **If p then q**

If you drink and drive, you get arrested.

All p are q

p implies q

q is necessary for p

p is sufficient for q

Drinking and driving is sufficient for getting arrested.

q if p

P only if q

You drink and drive only if you get arrested.

Variations of a Conditional Statement

p: "You earned a master's degree"
q: "You got a high-paying job"

Conditional	$p \rightarrow q$	If p, then q	"You earned a master's degree" → "You got a high-paying job"
Converse	$q \rightarrow p$	If q, then p	"You got a high-paying job" → "You earned a master's degree"
Inverse	$\sim p \rightarrow \sim q$	If not p, then not q	"You did not earn a master's degree" → "You did not get a high-paying job"
Contrapositive	$\sim q \rightarrow \sim p$	If not q, then not p	"You did not get a high-paying job" → "you did not earn a master's degree"

Equivalent!

Negation & Logical Equivalence

Logical Equivalence

- *Two statements are logically equivalent* statements when p $\leftrightarrow$ q is a tautology.
- Denoted by p $\equiv$ q
- In cases of compound statements, the two compound statements are logically equivalent <u>if and only if</u> they have the <u>same truth values</u> for <u>all possible combinations of truth values for the simple statements</u> that compose them.

Identifying Logically Equivalent Statements: Practice

Q: pre $p \rightarrow q$ and $\sim p \vee q$ logically equivalent?

How to show whether they are equivalent or not? If the truth values for both statements are **identical**, then they are logically equivalent.

p	q	**p → q**
T	T	**T**
T	F	**F**
F	T	**T**
F	F	**T**

p	q	~p	**~pV q**
T	T	F	**T**
T	F	F	**F**
F	T	T	**T**
F	F	T	**T**

Practice:

pre conditional and contrapositive are logically equivalent? Yes (The identical truth table proves it.)

Conditional
$p \rightarrow q$

Converse
$q \rightarrow p$

Inverse
$\sim p \rightarrow \sim q$

Contrapositive
$\sim q \rightarrow \sim p$

P	q	**$p \rightarrow q$**	~P	~q	**$\sim q \rightarrow \sim p$**
T	T	**T**	F	F	**T**
T	F	**F**	F	T	**T**
F	T	**T**	T	F	**F**
F	F	**T**	T	T	**T**

How about converse and inverse?

Testing Logical Equivalency

Let's see whether the following two statements are logically equivalent by completing the truth table.

~(p ^ q) Ξ ~p v ~q

De Morgan's Laws for Logic

p	q	p ^ q	~(p ^ q)	~p	~q	~p v ~q
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

Let's see whether the following two statements are logically equivalent by completing the truth table. ~(p v q) Ξ ~p ^ ~q

p	q	p v q	~(p v q)	~p	~q	~p ^ ~q

De Morgan’s Laws for Logic

- DeMorgan's law is a rule of inference pertaining to the NOT, AND, and OR operators. It is used to *distribute* a negative to a conjunction or disjunction. (Similar to laws of sets)

$$\sim(p \wedge q) \equiv \sim p \vee \sim q$$

“It is not true that he took both math and physics”

“He did not take math or he did not take Physics."

$$\sim(p \vee q) \equiv \sim p \wedge \sim q$$

"It is not true that the book is boring or the newspaper is interesting.”

"The book is not boring and the newspaper is not interesting."

Negation of Compound Statements

Statement	Negation	Equivalent Negation
p ^ q	**~(p ^ q)**	**~p V ~q**
pV q	**~(pV q)**	**~p ^ ~q**
p → q	**~(p→q)**	**p ^ ~q**

De Morgan's Laws

Negation of Conditional (Tricky!)

Ex. If I have money, then I will go to the movies.

How can we translate ~(p→q)?

The Negation of Conditional

Let p: I have money, and q: I will go to the movie

- p->q: “If I have money, then I will go to the movies.”
- the negation of p→q, i.e., ~(p→q):

“____________________????____________________________”

BE CAREFUL!
You might first think that the negation of p->q may be stated as follows: “If I don’t have money, then I will not go to the movies.”

But that’s NOT a correct translation!
To know why it isn’t, let’s re-translate the sentence in terms of “p” “q” again. It is “~p →~q.” Is this equivalent to “~(p→q)”? Compare their truth tables.

Practice

- Construct Truth Table

1) $\sim(p \rightarrow q)$

p	q		

2) $p \wedge \sim q$

p	q	~q	

Are 1) and 2) logically equivalent?

If they are equivalent, what would be the proper statement of $\sim(p \rightarrow q)$ when p represents "I have money" and q represents "I will go to the movies"?

p	q	~p	~q	

cf) "If I don't have money,
then I will not go to the movies." $(\sim p \rightarrow \sim q)$

Practice
Truth table with three variables

Construct the truth table for the following statement:

$p \wedge (q \rightarrow r)$

Treat $q \rightarrow r$ as one single variable, say, "**s**" whose truth values are stated in the 4th column. What you need to do is simply determine the truth values of conjunctive statement **p ^ s**!

How many rows do you need? What are the possible combinations of truth values of p, q, r? Each has 2 possible truth values, so 2x2x2=8

			s	p ^ s
p	**q**	**r**	**$q \rightarrow r$**	**$p \wedge (q \rightarrow r)$**
T	T	T		T
T	T	F		F
T				T
T				T
F				F
F				F
F				F
F	F	F		F

Practice
Truth Table with three variables

Construct the truth table for 1) p v (q ^ r) and 2) (p v q) ^ (p v r), then determine whether 1) and 2) are logically equivalent or not.

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

There must be 8 rows!

2^n (n= number of variables such as p, q, r…)

Are they logically equivalent?

Validity

Valid argument

- Definition

An argument is valid if the conclusion necessarily follows from the premise.

To be more specific..(in what way it is necessary??)

pn argument is logically valid whenever it's impossible for the premises to be true and the conclusion false at the same time. They are logically invalid whenever it is possible for the premises to be true and the conclusion to be false at the same time.

Valid: it is impossible for the conclusion to be false, when the premises are true.

We can better understand this if we remember the truth table of conditional:

NB

The validity of an argument is NOT about whether or not the conclusion is true.

The definition of valid argument only says that true premises and false conclusion is impossible! This implies that an argument can be valid when it has FpLSE PREMISES and TRUE rONrLUSION, and when its PREMISES are FpLSE and rONrLUSION is FpLSE as well.

p	q	p → q
T	T	T
T	F	F
F	T	T
F	F	T

- Recall the truth table for p→q.

We have seen that p→q is false only when p is true and (yet) q is false! If p were false, then p→q automatically becomes true.

- This is the same for the case of an argument. Since argumentation is a form of conditional (if *premises* then *conclusion*), we say that an argument becomes "invalid" only when it has TRUE premises and (yet) FALSE conclusion. (reflecting 2nd row of the truth table above)

Common Valid argument Forms

All p are q

<u>x is a p_____</u>

x is q

$p \rightarrow q$

$\underline{p \quad\quad\quad}$

q

p or q

<u>~p_____</u>

q

$p \rightarrow q$

$\underline{q \rightarrow r\ }$

$P \rightarrow r$

WHAT WOULD BE THE CONCLUSIONS, IF EACH ARGUMENT BELOW IS VALID?

$(p \land q) \rightarrow \sim r$	$(q \rightarrow r) \lor (p \rightarrow q)$	$(p \lor r) \rightarrow \sim r$
$\underline{r\qquad\qquad}$	$\underline{\sim(q \rightarrow r)\qquad\qquad}$	$\underline{r\qquad\qquad}$
?	?	?

***GIVEN THE PREMISES, THE CONCLUSION "MUST" BE ..**

- **In validity test, what matters is NOT whether the conclusion (and even the premises) is true; the main concern is whether the conclusion MUST follow from the premises.**

- Examples of valid argument (next page) show that a valid argument does not have to have "true" conclusion and "true" premises; the relation between the premises and conclusion matters!

Does conclusion necessarily follow from the premises, YES!! THEN WHY?

What valid forms does each argument have?

All organisms with wings can fly. F
Penguins have wings. T
Therefore, penguins can fly. F

The capital of Massachusetts is Springfield. F
No banks are located in the capital of Massachusetts. F
Therefore, Springfield has no banks. F

The capital of Massachusetts is Springfield.
Springfield is the home of the Boston Red Sox.
Therefore, the capital of Massachusetts is the home of the Boston Red Sox.

The capital of Massachusetts is Boston. T
Boston is the home of the Boston Red Sox. T
Therefore, the capital of Massachusetts is the home of the Boston Red Sox. T

Either $2+3=4$ or $3x2=5$.
It is not the case that $2+3=4$
Therefore, $3x2=5$.

Formal Fallacies $\rightarrow$ not valid!

Conclusions do NOT necessarily follow from the premises.

- Fallacy of inverse

$$\begin{array}{l} p \rightarrow q \\ \underline{\sim p \qquad} \\ \sim q \end{array}$$

If it rains then it is cloudy.
It is not raining.
Therefore, it is not cloudy.

- Fallacy of converse

$$\begin{array}{l} p \rightarrow q \\ \underline{q \qquad} \\ p \end{array}$$

If it rains then it is cloudy.
It is cloudy.
Therefore, it is raining.

$$\boxed{\begin{array}{l} p \rightarrow q \\ \underline{p \qquad} \\ q \qquad \text{valid} \end{array}}$$

- Fallacy of the inclusive "or"

$$\begin{array}{l} p \text{ or } q \\ \underline{p \qquad} \\ \sim q \end{array}$$

Either I am in the library or I am studying.
I am in the library.
Therefore, I am not studying. (!!!???)

$$\boxed{\begin{array}{l} p \text{ or } q \\ \underline{\sim p \qquad} \\ q \qquad \text{valid} \end{array}}$$

Validity Test through Truth Tables

- Step 1: Translate the argument into logical statement (use variables and connectives).
- Step 2: Write the argument as a conditional statement; first, make a *conjunction of "all" premises* (antecedent) and then connect it with the conclusion (consequence) using "→."
- Step 3: set up a truth table

Statements	Premise ^ premise → conclusion

- *Determining Validity:*

  If all truth values under → are Ts, then the argument is valid; otherwise, invalid.

Practice

Argument 1

Socrates is an animal.

If Socrates is a mammal, then Socrates is an animal.

Therefore, Socrates is a mammal.

Step1

p

<u>q → p</u>

q

Each letter stands for a specific statement:

p: Socrates is an animal.

q: Socrates is a mammal.

Step 2 - 4

Conjunction of *all* premises

Conditional

p	q	q → p	p ^ (q →p)	p ^ (q →p) ➔ q
T	T	T	T	T
T	F	T	T	F
F	T	F	F	T
F	F	T	F	T

INVALID argument: not all the truth values of the conditional statement in the last column are true

Practice

Step 1

$p \rightarrow q$
$\underline{\sim q \quad}$
$\sim p$

Argument 2:

If a figure has three sides, then it is a triangle.

This figure is not a triangle.

Therefore, this figure does not have three sides.

VALID argument: all the truth values of the conditional statement the last column are true

Conjunction of *all* premises

Step 2 - 4

p	**q**	**~p**	**~q**	**p→q**	**(p→q) ^ ~q**	**[(p→q)^~q] → ~p**
T	T	F	F	T	F	**T**
T	F	F	T	F	F	**T**
F	T	T	F	T	F	**T**
F	F	T	T	T	T	**T**

Conditional