Solving the membership problem for certain subgroups of $SL_2(\mathbb{Z})$ Sandie Han, Ariane Masuda, Satyanand Singh, and Johann Thiel

Abstract

For positive integers u and v, let $L_u = \begin{bmatrix} 1 & 0 \\ u & 1 \end{bmatrix}$ and $R_v = \begin{bmatrix} 1 & v \\ 0 & 1 \end{bmatrix}$. Let $G_{u,v}$ be the group generated by L_u and R_v . The membership problem for $G_{u,v}$ asks the following question: given a 2-by-2 matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, is there a relatively straightforward method for determining if \overline{M} is a member of $G_{u,v}$? In the case where u = 2 and v = 2, Sanov was able to show that simply checking some divisibility conditions for a, b, c, and d is enough to make this determination. In a previous paper, the authors answered this question by finding a characterization of matrices M in $G_{u,v}$ when $u, v \geq 3$ in terms of the short continued fraction representation of b/d. By modifying our previous work, we are able to extend our previous result to the case where $u, v \ge 2$ with $uv \neq 4.$

Background

For positive integers u and v, let $L_u = \begin{bmatrix} 1 & 0 \\ u & 1 \end{bmatrix}$, $R_v = \begin{bmatrix} 1 & v \\ 0 & 1 \end{bmatrix}$, and $G_{u,v}$ be the group generated by L_u and R_v . Furthermore, using the notation from [1], let

 $\mathscr{G}_{u,v} = \left\{ \begin{bmatrix} 1 + uvn_1 & vn_2 \\ un_3 & 1 + uvn_4 \end{bmatrix} \in SL_2(\mathbb{Z}) : (n_1, n_2, n_3, n_4) \in \mathbb{Z}^4 \right\}.$

Note that $\mathscr{G}_{u,v}$ is a group and that $G_{u,v} \subseteq \mathscr{G}_{u,v}$ when $u, v \geq 2$ [2, Proposition 1.1].

Given a rational number q, if there exist integers q_0, q_1, \ldots, q_r (referred to as partial quotients) such that

$$q = q_0 + \frac{1}{q_1 + \frac{1}{q_2 + \cdots + \frac{1}{q_2}}},$$

then we refer to such an identity as a continued fraction representation of q and denote it by $[q_0, q_1, \ldots, q_r]$. We refer to the unique such representation where $q_i \ge 1$ for 0 < i < r and $q_r > 1$ for r > 0 as the short continued fraction representation of q.

In [1], Esbelin and Gutan gave the following clear characterization of members of $G_{u,v}$ in terms of related continued fraction representations.

Theorem 1 (Esbelin and Gutan [1]) Suppose that $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathscr{G}_{k,k}$ for some $k \geq 2$. Then $M \in G_{k,k}$ if and only if at least one of the rationals c/a and b/d has a continued fraction expansion having all partial quotients in $k\mathbb{Z}$.

In [2], we showed that Theorem 1 could be modified and written in terms of the short continued fraction representations of either c/a or b/d, when $u, v \geq 3$. In particular, we developed a simple algorithm that, when applied to the short continued fraction representation of b/d, determines whether or not the sought after continued fraction expansion in Theorem 1 exists.

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Preliminaries

$\begin{split} & \text{Let } A = \bigcup_{i=n}^{\infty} (\mathbb{Z} \times \mathbb{Z}_{q_0}^r). \text{ We denote an element of } A \text{ by } [q_0, q_1, q_1 - [q_0, q_1, q_2, \ldots, q_r]] := [[-q_0, -q_1, -q_2, \ldots, -q_r]]. \\ & \text{For any nonnegative integers } m \text{ and } n, \text{ let} \\ & [q_0, q_1, q_2, \ldots, q_m] \oplus [[p_0, p_1, p_2, \ldots, p_n]] := \begin{cases} [[q_0, q_1, q_2, \ldots, q_m, p_0] & \text{if } p_0 \neq 0, \\ [[q_0, q_1, q_2, \ldots, q_m] \oplus [[p_0, q_1, p_2, \ldots, p_n]] & \text{if } p_0 \neq 0, \\ [[q_0, q_1, q_2, \ldots, q_m] \oplus [[p_0, q_1, q_2, \ldots, q_n]] \in A : [q_i, \ldots, q_n] \neq 0 \text{ when } 0 < A_1 = \{[q_0, q_1, q_2, \ldots, q_n] \oplus A : [q_i, \ldots, q_n] \neq 0 \text{ when } 0 < i < r, and q_r > 1 \text{ when } 0 < i < r, and q_r > 1 \text{ when } 0 < i < r \end{cases} \\ & \text{Define the function } C : \mathbb{Q} \to A_1 \text{ by} \\ & C(x) = [[x_0, x_1, x_2, \ldots, x_r]] \\ & \text{if } [x_0, x_1, x_2, \ldots, q_r] \in A \text{ satisfies the } (u, v) \text{-divisibility propert} \\ & \text{even and } u[q_i, whn] is \text{ obd}. \\ & \text{Define the function } C : \mathbb{Q} \to A_1 \text{ by} \\ & C(x) = [[x_0, x_1, x_2, \ldots, x_r]] \\ & \text{if } [[q_0, q_1, q_2, \ldots, q_r]] \\ & = \begin{cases} [q_0, q_1, q_2, \ldots, q_r] \in A \text{ satisfies the } (u, v) \text{-divisibility propert} \\ & q_0, q_1, q_2, \ldots, q_r] \end{bmatrix} \\ & f_{n,v}([[q_0, q_1, q_2, \ldots, q_r]]) \\ & \text{if } v \nmid q_0, q_1 = 2 \\ [q_0 + 1] \oplus f_{v,a}([[q_2, q_2 + 1, q_3, \ldots, q_r]]) & \text{if } v \nmid q_0, q_1 = 2 \\ [q_0 + 1] \oplus f_{v,a}([[q_2, q_2, \ldots, q_r]]) \\ & \text{if } v \restriction q_0, q_1 = 2 \\ [q_0 + 1] \oplus f_{v,a}([[q_2, q_2, \ldots, q_r]]) & \text{otherwise.} \end{cases} \\ \\ & \text{Define } g_{u,v} : A_2 \to A_1 \text{ recursively by} \\ \end{cases} \\ \begin{array}{c} g_{a,v}([[q_0, q_1, q_2, \ldots, q_r]]) \\ & \text{if either } q_1 < 0 \text{ and } u \neq 2, \text{ or } q_1 < -2 \text{ and } u = 2 \\ [q_0 - 1, 2] \oplus g_{v,u}([[q_1 - 1, q_3, \ldots, q_r]]) \\ & \text{if } q_1 = -2, r > 1, \text{ and } u = 2, \\ [q_0 - 1, 2] \\ & \text{if } q_1 = -2, r = 1, \text{ and } u = 2, \\ [q_0 \oplus g_{v,u}([[q_1, q_2, \ldots, q_r]]) \\ \end{array} \end{array}$		
$ \begin{split} & \text{For any nonnegative integers } m \text{ and } n, \text{ let} \\ & \llbracket q_0, q_1, q_2, \dots, q_m \rrbracket \oplus \llbracket p_0, p_1, p_2, \dots, p_n \rrbracket := \begin{cases} \llbracket q_0, q_1, q_2, \dots, q_m, p_0 \\ & \Pi p_0 \neq 0, \\ \llbracket q_0, q_1, q_2, \dots, q_m \rrbracket \in A : \llbracket q_0, q_1, q_2, \dots, q_m \neq n \\ & \text{otherwisse} \end{cases} \\ & \text{Let} \\ & A_0 = \{ \llbracket q_0, q_1, q_2, \dots, q_n \rrbracket \in A : [q_i, \dots, q_r] \neq 0 \text{ when } 0 < \\ & A_1 = \{ \llbracket q_0, q_1, q_2, \dots, q_r \rrbracket \in A : [q_i, \dots, q_r] \neq 0 \text{ when } 0 < \\ & A_1 = \{ \llbracket q_0, q_1, q_2, \dots, q_r \rrbracket \in A_0 : q_i > 1 \text{ when } n > 0 \}, \\ & A_2 = \{ \llbracket q_0, q_1, q_2, \dots, q_r \rrbracket \in A_0 : q_i > 1 \text{ when } n < i \leq r \end{cases} \\ & \text{Define the function } C : \mathbb{Q} \to A_1 \text{ by} \\ & C(x) = \llbracket x_0, x_1, x_2, \dots, x_r \rrbracket \\ & \text{if } \llbracket x_0, x_1, x_2, \dots, x_r \rrbracket \text{ is the short continued fraction representation that } \llbracket q_0, q_1, q_2, \dots, q_r \rrbracket \in A \text{ satisfies the } (u, v) \text{-divisibility propert even and } u [q_i \text{ when } i \text{ is odd.} \\ & \text{Define } f_{n,v} : (A_1 \to A_2 \text{ recursively by} \end{cases} \\ & f_{u,v} (\llbracket q_0, q_1, q_2, \dots, q_r \rrbracket) \\ & = \begin{cases} \llbracket q_0 \parallel & \text{if } r = 0, \\ \llbracket q_0 + 1 \rrbracket \oplus -f_{v,u}(\llbracket q_2 + 1, q_3, \dots, q_r \rrbracket) & \text{if } v \nmid q_0, q_1 = 2 \\ \llbracket q_0 + 1 \rrbracket \oplus f_{v,u}(\llbracket q_1, q_2, \dots, q_r \rrbracket) & \text{if } v \nmid q_0, q_1 = 2 \\ \llbracket q_0 + 1 \rrbracket \oplus f_{v,u}(\llbracket q_1, q_2, \dots, q_r \rrbracket) & \text{otherwise.} \end{cases} \\ & \text{Define } g_{u,v} : A_2 \to A_1 \text{ recursively by} \end{cases} \\ & f_{u,v} (\llbracket q_0, q_1, q_2, \dots, q_r \rrbracket) \\ & = \begin{cases} \llbracket q_0 \parallel & \ \phi = f_{v,u}(\llbracket q_1, q_2, \dots, q_r \rrbracket) & \text{otherwise.} \\ & \text{Define } g_{u,v} : A_2 \to A_1 \text{ recursively by} \end{cases} \\ & f_{u,0} \parallel & \ q_0 - 1, 1 \rrbracket \oplus g_{v,u}(-\llbracket q_1 + 1, q_2, \dots, q_r \rrbracket) \\ & \text{if either } q_1 < 0 \text{ and } u \neq 2, \text{ or } q_1 < -2 \text{ and } u = \\ & \llbracket q_0 = 1, 2 \rrbracket \oplus g_{v,u}(\llbracket q_2 - 1, q_3, \dots, q_r \rrbracket) \\ & \text{if } either q_1 < 0 \text{ and } u = 2, \end{cases} \end{cases}$	Let $A = \bigcup_{r=0}^{\infty} (\mathbb{Z} \times \mathbb{Z}_{\neq 0}^{r})$. We denote an element	of A by $\llbracket q_0, q_1, q_2$
$ \begin{split} & \llbracket q_0, q_1, q_2, \dots, q_m \rrbracket \odot \llbracket p_0, p_1, p_2, \dots, p_n \rrbracket := \begin{cases} \llbracket q_0, q_1, q_2, \dots, q_m \rrbracket \odot \llbracket p_0 \neq 0, \\ \llbracket q_0, q_1, q_2, \dots, q_m \rrbracket \in A; \\ \llbracket q_0, q_1, q_2, \dots, q_r \rrbracket \in A : \begin{bmatrix} q_i, \dots, q_r \end{bmatrix} \neq 0 \text{ when } 0 < \\ & A_1 = \{\llbracket q_0, q_1, q_2, \dots, q_r \rrbracket \in A_0 : q_i \geq 1 \\ & \text{when } 0 < i < r, \text{ and } q_r > 1 \text{ when } r > 0 \}, \\ & A_2 = \{\llbracket q_0, q_1, q_2, \dots, q_r \rrbracket \in A_0 : \ q_i\ > 1 \text{ when } 0 < i \leq r \end{cases} \\ \text{Define the function } C : \mathbb{Q} \to A_1 \text{ by} \\ & C(x) = \llbracket x_0, x_1, x_2, \dots, x_r \rrbracket \\ \text{if } [x_0, x_1, x_2, \dots, x_r] \text{ is the short continued fraction representation that } [q_0, q_1, q_2, \dots, q_r] \in A \text{ satisfies the } (u, v) \text{-divisibility properterven and } u \ q_i \text{ when } i \text{ is odd.} \\ \text{Define } f_{u,v} : \ q_1 - f_{v,u}(\llbracket q_2 + 1, q_3, \dots, q_r]) & \text{ if } v \nmid q_0 \text{ and } q_1 \\ & \llbracket q_0 + 1 \rrbracket \oplus f_{v,u}(\llbracket q_2, \dots, q_r] \end{bmatrix} \\ = \begin{cases} \llbracket q_0 & \ f v + q_0\} & \text{ if } v \nmid q_0, q_1 = 2 \\ \llbracket q_0 + 1, -2 \rrbracket & \text{ if } v \nmid q_0, q_1 = 2 \\ \llbracket q_0 + 1, -2 \rrbracket & \text{ if } v \nmid q_0, q_1 = 2 \\ \llbracket q_0 & \ g_0 + f_{v,u}(\llbracket q_1, q_2, \dots, q_r] \end{bmatrix} \\ \text{Define } g_{u,v} : A_2 \to A_1 \text{ recursively by} \end{cases} \\ \text{Define } g_{u,v} : \begin{bmatrix} q_0, q_1, q_2, \dots, q_r \rrbracket \end{bmatrix} \\ = \begin{cases} \llbracket q_0 & \ g_0 - 1, 1 \rrbracket \oplus g_{v,u}(-\llbracket q_1 + 1, q_2, \dots, q_r] \\ \parallel q_0 - 1, 2 \rrbracket \oplus g_{v,u}(\llbracket q_2 - 1, q_3, \dots, q_r] \end{bmatrix} \\ \text{ if either } q_1 < 0 \text{ and } u = 2, \end{cases} \end{aligned}$		
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$ \begin{array}{l} \mbox{Let} \\ A_0 = \left\{ [q_0, q_1, q_2, \dots, q_r] \in A : [q_i, \dots, q_r] \neq 0 \text{ when } 0 < \\ A_1 = \left\{ [q_0, q_1, q_2, \dots, q_r] \in A_0 : q_i \geq 1 \\ & \mbox{when } 0 < i < r, \mbox{ and } q_r > 1 \mbox{ when } r > 0 \right\}, \\ A_2 = \left\{ [q_0, q_1, q_2, \dots, q_r] \in A_0 : q_i > 1 \mbox{ when } 0 < i \leq r \end{array} \right. \\ \mbox{Define the function } C : \mathbb{Q} \to A_1 \mbox{ by } \\ C(x) = [[x_0, x_1, x_2, \dots, x_r]] \mbox{ if } r = 0, \\ [[x_0, x_1, x_2, \dots, q_r]] \in A \mbox{ satisfies the } (u, v) \mbox{-divisibility propert} \mbox{ even and } u q_i \mbox{ when } i \mbox{ is odd.} \\ \mbox{Define } f_{u,v} : [x_1 \to A_2 \mbox{ recursively by } \end{array} \right. \\ \left. \begin{array}{l} f_{u,v}([[q_0, q_1, q_2, \dots, q_r]]) & \mbox{ if } r = 0, \\ [[q_0] = f_{u,v} : A_1 \to A_2 \mbox{ recursively by } \end{array} \right. \\ \left. \begin{array}{l} f_{u,v}([[q_0, q_1, q_2, \dots, q_r]]) & \mbox{ if } v \nmid q_0 \mbox{ and } q_1 \\ [[q_0] + 1] \oplus -f_{v,u}([[q_2 + 1, q_3, \dots, q_r]]) & \mbox{ if } v \nmid q_0, q_1 = 2 \\ [[q_0] + 1] \oplus f_{v,u}([[q_2, q_2 + 1, q_3, \dots, q_r]]) & \mbox{ if } v \restriction q_0, q_1 = 2 \\ [[q_0] = f_{v,u}([[q_1, q_2, \dots, q_r]]) & \mbox{ otherwise.} \end{array} \right. \\ \mbox{Define } g_{u,v} : A_2 \to A_1 \mbox{ recursively by } \end{array} \right. \\ \\ \mbox{Define } g_{u,v} ([[q_0, q_1, q_2, \dots, q_r]]) & \mbox{ if either } q_1 < 0 \mbox{ and } u \neq 2, \mbox{ or } q_1 < -2 \mbox{ and } u = 2 \\ [[q_0] = f_{v,u}([[q_0 - 1, 1] \oplus g_{v,u}(-[[q_1 + 1, q_2, \dots, q_r]]) & \mbox{ if } e_1 = -2, r > 1, \mbox{ and } u = 2, \end{array} \right.$	$\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \bigoplus \begin{bmatrix} n_1 & n_2 & \cdots & n_n \end{bmatrix} := $	if $p_0 \neq 0$,
$ \begin{array}{l} \mbox{Let} \\ A_0 = \left\{ [q_0, q_1, q_2, \dots, q_r] \in A : [q_i, \dots, q_r] \neq 0 \text{ when } 0 < \\ A_1 = \left\{ [q_0, q_1, q_2, \dots, q_r] \in A_0 : q_i \geq 1 \\ & \mbox{when } 0 < i < r, \mbox{ and } q_r > 1 \mbox{ when } r > 0 \right\}, \\ A_2 = \left\{ [q_0, q_1, q_2, \dots, q_r] \in A_0 : q_i > 1 \mbox{ when } 0 < i \leq r \end{array} \right. \\ \mbox{Define the function } C : \mathbb{Q} \to A_1 \mbox{ by } \\ C(x) = [[x_0, x_1, x_2, \dots, x_r]] \mbox{ if } r = 0, \\ [[x_0, x_1, x_2, \dots, q_r]] \in A \mbox{ satisfies the } (u, v) \mbox{-divisibility propert} \mbox{ even and } u q_i \mbox{ when } i \mbox{ is odd.} \\ \mbox{Define } f_{u,v} : [x_1 \to A_2 \mbox{ recursively by } \end{array} \right. \\ \left. \begin{array}{l} f_{u,v}([[q_0, q_1, q_2, \dots, q_r]]) & \mbox{ if } r = 0, \\ [[q_0] = f_{u,v} : A_1 \to A_2 \mbox{ recursively by } \end{array} \right. \\ \left. \begin{array}{l} f_{u,v}([[q_0, q_1, q_2, \dots, q_r]]) & \mbox{ if } v \nmid q_0 \mbox{ and } q_1 \\ [[q_0] + 1] \oplus -f_{v,u}([[q_2 + 1, q_3, \dots, q_r]]) & \mbox{ if } v \nmid q_0, q_1 = 2 \\ [[q_0] + 1] \oplus f_{v,u}([[q_2, q_2 + 1, q_3, \dots, q_r]]) & \mbox{ if } v \restriction q_0, q_1 = 2 \\ [[q_0] = f_{v,u}([[q_1, q_2, \dots, q_r]]) & \mbox{ otherwise.} \end{array} \right. \\ \mbox{Define } g_{u,v} : A_2 \to A_1 \mbox{ recursively by } \end{array} \right. \\ \\ \mbox{Define } g_{u,v} ([[q_0, q_1, q_2, \dots, q_r]]) & \mbox{ if either } q_1 < 0 \mbox{ and } u \neq 2, \mbox{ or } q_1 < -2 \mbox{ and } u = 2 \\ [[q_0] = f_{v,u}([[q_0 - 1, 1] \oplus g_{v,u}(-[[q_1 + 1, q_2, \dots, q_r]]) & \mbox{ if } e_1 = -2, r > 1, \mbox{ and } u = 2, \end{array} \right.$	$\llbracket q_0, q_1, q_2, \ldots, q_m \rrbracket \oplus \llbracket p_0, p_1, p_2, \ldots, p_n \rrbracket \cdot - \left\{ \llbracket q_0, q_0, q_0, q_0, q_0, q_0, q_0, q_0,$	$q_1, q_2, \ldots, q_m + q_m$
$ \begin{array}{l} \mbox{Let} & A_0 = \{ \ q_0, q_1, q_2, \dots, q_r \ \in A : [q_i, \dots, q_r] \neq 0 \mbox{ when } 0 < A_1 = \{ \ q_0, q_1, q_2, \dots, q_r \ \in A_0 : q_i \geq 1 \\ & \mbox{ when } 0 < i < r, \mbox{ and } q_r > 1 \mbox{ when } r > 0 \}, \\ A_2 = \{ \ q_0, q_1, q_2, \dots, q_r \ \in A_0 : q_i > 1 \mbox{ when } 0 < i \leq r \\ \mbox{Define the function } C : \mathbb{Q} \to A_1 \mbox{ by } \\ & C(x) = \ x_0, x_1, x_2, \dots, x_r \ \\ \mbox{if } [x_0, x_1, x_2, \dots, x_r] \mbox{ is the short continued fraction representati} \\ \mbox{that } [q_0, q_1, q_2, \dots, q_r] \in A \mbox{ satisfies the } (u, v) \mbox{-divisibility propert} \\ \mbox{even and } u q_i \mbox{ when } i \mbox{ is odd.} \\ \mbox{Define } f_{u,v} : \ \Phi - f_{v,u}([q_2 + 1, q_3, \dots, q_r]]) \mbox{ if } v \nmid q_0 \mbox{ and } q_1 \\ \\ [q_0 + 1] \oplus - f_{v,u}([q_2 + 1, q_3, \dots, q_r]]) \mbox{ if } v \nmid q_0, q_1 = 2 \\ \\ [q_0 + 1] \oplus f_{v,u}([q_1, q_2, \dots, q_r]]) \mbox{ if } v \restriction q_0, q_1 = 2 \\ \\ [q_0 + 1, -2] \mbox{ if } v \restriction q_0, q_1 = 2 \\ \\ [q_0 + 1, -2] \mbox{ if } v \restriction q_0, q_1 = 2 \\ \\ [q_0 - 1, 1] \oplus g_{v,u}(-[[q_1 + 1, q_2, \dots, q_r]]) \mbox{ if } v \restriction q_0, q_1 = 2 \\ \\ g_{u,v}([[q_0, q_1, q_2, \dots, q_r]]) \mbox{ if either } q_1 < 0 \mbox{ and } u \neq 2, \mbox{ or } q_1 < -2 \mbox{ and } u = 2 \\ \\ \ q_0 - 1, 2\ \oplus g_{v,u}([[q_2 - 1, q_3, \dots, q_r]]) \mbox{ if } q_1 = -2, r > 1, \mbox{ and } u = 2, \\ \end{array}$		otherwise
$ \begin{array}{l} A_{1} = \left\{ [q_{0},q_{1},q_{2},\ldots,q_{r}] \in A_{0}:q_{i} \geq 1 \\ & \text{when } 0 < i < r, \text{ and } q_{r} > 1 \text{ when } r > 0 \right\}, \\ A_{2} = \left\{ [q_{0},q_{1},q_{2},\ldots,q_{r}] \in A_{0}: q_{i} > 1 \text{ when } 0 < i \leq r \right\} \\ \text{Define the function } C: \mathbb{Q} \to A_{1} \text{ by} \\ & C(x) = [x_{0},x_{1},x_{2},\ldots,x_{r}] \\ \text{if } [x_{0},x_{1},x_{2},\ldots,x_{r}] \text{ is the short continued fraction representati} \\ \text{that } [q_{0},q_{1},q_{2},\ldots,q_{r}] \in A \text{ satisfies the } (u,v) \text{-divisibility propert} \\ \text{even and } u q_{i} \text{ when } i \text{ is odd.} \\ \text{Define } f_{u,v}: A_{1} \to A_{2} \text{ recursively by} \\ \\ f_{u,v}([q_{0},q_{1},q_{2},\ldots,q_{r}])) \\ = \begin{cases} [q_{0}] & \text{if } r = 0, \\ [q_{0}] + 1] \oplus -f_{v,u}([q_{2}+1,q_{3},\ldots,q_{r}]) & \text{if } v \nmid q_{0} \text{ and } q_{1} \\ [q_{0}+1] \oplus f_{v,u}([-2,q_{2}+1,q_{3},\ldots,q_{r}]) & \text{if } v \nmid q_{0}, q_{1} = 2 \\ [q_{0}+1,-2] & \text{if } v \nmid q_{0}, q_{1} = 2 \\ [q_{0}] \oplus f_{v,u}([q_{1},q_{2},\ldots,q_{r}]) & \text{otherwise.} \\ \end{array} \\ \text{Define } g_{u,v}: A_{2} \to A_{1} \text{ recursively by} \\ \\ g_{u,v}([q_{0},q_{1},q_{2},\ldots,q_{r}]) & \text{if either } q_{1} < 0 \text{ and } u \neq 2, \text{ or } q_{1} < -2 \text{ and } u = 2 \\ [q_{0}-1,2] \oplus g_{v,u}([q_{2}-1,q_{3},\ldots,q_{r}]) & \text{if } q_{1}=-2, r > 1, \text{ and } u = 2, \end{cases} $	<u>_</u>	
$\begin{split} & \text{when } 0 < i < r, \text{ and } q_r > 1 \text{ when } r > 0 \}, \\ A_2 &= \{ [q_0, q_1, q_2, \dots, q_r] \in A_0 : q_i > 1 \text{ when } 0 < i \leq r \\ \text{Define the function } C : \mathbb{Q} \to A_1 \text{ by} \\ & C(x) &= [x_0, x_1, x_2, \dots, x_r] \\ \text{if } [x_0, x_1, x_2, \dots, x_r] \text{ is the short continued fraction representation that } [q_0, q_1, q_2, \dots, q_r] \in A \text{ satisfies the } (u, v) \text{-divisibility propert even and } u q_i \text{ when } i \text{ is odd.} \\ \text{Define } f_{u,v} : A_1 \to A_2 \text{ recursively by} \\ & f_{u,v} ([q_0, q_1, q_2, \dots, q_r]]) \\ &= \begin{cases} [q_0] & \text{if } r = 0, \\ [q_0 + 1] \oplus -f_{v,u} ([q_2 + 1, q_3, \dots, q_r]]) & \text{if } v \nmid q_0 \text{ and } q_1 \\ [q_0 + 1] \oplus f_{v,u} ([-2, q_2 + 1, q_3, \dots, q_r]]) & \text{if } v \nmid q_0, q_1 = 2 \\ [q_0] \oplus f_{v,u} ([q_1, q_2, \dots, q_r]]) & \text{otherwise.} \end{cases} \\ \text{Define } g_{u,v} : A_2 \to A_1 \text{ recursively by} \\ & \\ \text{Define } g_{u,v} : A_2 \to A_1 \text{ recursively by} \\ & \\ g_{u,v} ([q_0, q_1, q_2, \dots, q_r]]) & \text{if either } q_1 < 0 \text{ and } u \neq 2, \text{ or } q_1 < -2 \text{ and } u = 2 \\ [q_0 - 1, 2] \oplus g_{v,u} ([q_2 - 1, q_3, \dots, q_r]]) & \text{if } q_1 = -2, r > 1, \text{ and } u = 2, \end{cases} \end{aligned}$		$r] \neq 0$ when $0 < $
$\begin{split} A_2 &= \{ [\![q_0,q_1,q_2,\ldots,q_r]\!] \in A_0: q_i > 1 \text{ when } 0 < i \leq r \\ \text{Define the function } C: \mathbb{Q} \to A_1 \text{ by} \\ & C(x) = [\![x_0,x_1,x_2,\ldots,x_r]\!] \\ \text{if } [x_0,x_1,x_2,\ldots,x_r] \text{ is the short continued fraction representation that } [\![q_0,q_1,q_2,\ldots,q_r]\!] \in A \text{ satisfies the } (u,v) \text{-divisibility properterven and } u q_i \text{ when } i \text{ is odd.} \\ \text{Define } f_{u,v}: A_1 \to A_2 \text{ recursively by} \\ & f_{u,v}([\![q_0,q_1,q_2,\ldots,q_r]\!]) \\ &= \begin{cases} [\![q_0]\!] & \text{if } r = 0, \\ [\![q_0+1]\!] \oplus -f_{v,u}([\![q_2+1,q_3,\ldots,q_r]\!]) & \text{if } v \nmid q_0 \text{ and } q_1 \\ [\![q_0+1]\!] \oplus f_{v,u}([\![-2,q_2+1,q_3,\ldots,q_r]\!]) & \text{if } v \nmid q_0, q_1 = 2 \\ [\![q_0]\!] [\![q_0+1,-2]\!] & \text{if } v \nmid q_0, q_1 = 2 \\ [\![q_0]\!] \oplus f_{v,u}([\![q_1,q_2,\ldots,q_r]\!]) & \text{otherwise.} \end{cases} \\ \text{Define } g_{u,v}: A_2 \to A_1 \text{ recursively by} \\ \\ \text{Define } g_{u,v}: A_2 \to A_1 \text{ recursively by} \\ \\ g_{u,v}([\![q_0,q_1,q_2,\ldots,q_r]\!]) & \text{if either } q_1 < 0 \text{ and } u \neq 2, \text{ or } q_1 < -2 \text{ and } u = 2 \\ [\![q_0-1,2]\!] \oplus g_{v,u}([\![q_2-1,q_3,\ldots,q_r]\!]) & \text{if } q_1 = -2, r > 1, \text{ and } u = 2, \end{cases} $		1 b a.a. > 0]
$ \begin{array}{l} \text{Define the function } C: \mathbb{Q} \to A_1 \text{ by} \\ C(x) = [x_0, x_1, x_2, \ldots, x_r] \\ \text{if } [x_0, x_1, x_2, \ldots, x_r] \text{ is the short continued fraction representation that } [q_0, q_1, q_2, \ldots, q_r] \in A \text{ satisfies the } (u, v) \text{-divisibility properties even and } u q_i \text{ when } i \text{ is odd.} \\ \text{Define } f_{u,v}: A_1 \to A_2 \text{ recursively by} \\ \\ & \int [[q_0]] & \text{if } r = 0, \\ [[q_0] + 1] \oplus -f_{r,u}([q_2 + 1, q_3, \ldots, q_r]) & \text{if } v \nmid q_0 \text{ and } q_1 \\ [[q_0 + 1]] \oplus f_{v,u}([-2, q_2 + 1, q_3, \ldots, q_r]]) & \text{if } v \nmid q_0, q_1 = 2 \\ [[q_0] + 1, -2] & \text{if } v \nmid q_0, q_1 = 2 \\ [[q_0] \oplus f_{v,u}([q_1, q_2, \ldots, q_r]]) & \text{otherwise.} \end{array} \right. \\ \\ \text{Define } g_{u,v}: A_2 \to A_1 \text{ recursively by} \\ \\ \\ \begin{array}{l} g_{u,v}([[q_0, q_1, q_2, \ldots, q_r]]) & \text{if } v \nmid q_0, q_1 = 2 \\ [[q_0] = [q_0] \oplus f_{v,u}([q_1 + 1, q_2, \ldots, q_r]]) & \text{otherwise.} \end{array} \right. \\ \\ \text{Define } g_{u,v}: A_2 \to A_1 \text{ recursively by} \\ \\ \end{array} $		2
$C(x) = [\![x_0, x_1, x_2, \dots, x_r]\!]$ if $[x_0, x_1, x_2, \dots, x_r]$ is the short continued fraction representation that $[\![q_0, q_1, q_2, \dots, q_r]\!] \in A$ satisfies the (u, v) -divisibility properties even and $u q_i$ when i is odd. Define $f_{u,v} : A_1 \to A_2$ recursively by $\begin{cases} [\![q_0]\!] & \text{if } r = 0, \\ [\![q_0 + 1]\!] \oplus -f_{v,u}([\![q_2 + 1, q_3, \dots, q_r]\!]) & \text{if } v \nmid q_0 \text{ and } q_1 \\ [\![q_0 + 1]\!] \oplus f_{v,u}([\![-2, q_2 + 1, q_3, \dots, q_r]\!]) & \text{if } v \nmid q_0, q_1 = 2 \\ [\![q_0]\!] \oplus f_{v,u}([\![q_1, q_2, \dots, q_r]\!]) & \text{otherwise.} \end{cases}$ Define $g_{u,v} : A_2 \to A_1$ recursively by $g_{u,v}([\![q_0, q_1, q_2, \dots, q_r]\!]) & \text{otherwise.} \\ for all examples for all explicit events and u \neq 2, or q_1 < -2 and u = 2, [\![q_0 - 1, 2]\!] \oplus g_{v,u}([\![q_2 - 1, q_3, \dots, q_r]\!]) & \text{if } q_1 = -2, r > 1, and u = 2,$		
$\begin{split} & \text{if } [x_0, x_1, x_2, \dots, x_r] \text{ is the short continued fraction representati that } [q_0, q_1, q_2, \dots, q_r]] \in A \text{ satisfies the } (u, v) \text{-divisibility propert even and } u q_i \text{ when } i \text{ is odd.} \\ & \text{Define } f_{u,v} : A_1 \to A_2 \text{ recursively by} \\ & f_{u,v}(\llbracket q_0, q_1, q_2, \dots, q_r] \rrbracket) \\ & = \begin{cases} \llbracket q_0 \rrbracket & \text{if } r = 0, \\ \llbracket q_0 + 1 \rrbracket \oplus -f_{v,u}(\llbracket q_2 + 1, q_3, \dots, q_r] \rrbracket) & \text{if } v \nmid q_0 \text{ and } q_1 \\ \llbracket q_0 + 1 \rrbracket \oplus f_{v,u}(\llbracket -2, q_2 + 1, q_3, \dots, q_r] \rrbracket) & \text{if } v \nmid q_0, q_1 = 2 \\ \llbracket q_0 \rrbracket + 1, -2 \rrbracket & \text{if } v \nmid q_0, q_1 = 2 \\ \llbracket q_0 \rrbracket \oplus f_{v,u}(\llbracket q_1, q_2, \dots, q_r] \rrbracket) & \text{otherwise.} \end{cases} \\ & \text{Define } g_{u,v} : A_2 \to A_1 \text{ recursively by} \end{aligned}$		r]
$ \begin{aligned} & \text{that } [\![q_0, q_1, q_2, \dots, q_r]\!] \in A \text{ satisfies the } (u, v) \text{-divisibility propert} \\ & \text{even and } u q_i \text{ when } i \text{ is odd.} \\ & \text{Define } f_{u,v} : A_1 \to A_2 \text{ recursively by} \\ & f_{u,v} ([\![q_0, q_1, q_2, \dots, q_r]\!]) \\ & = \begin{cases} [\![q_0]\!] & \text{if } r = 0, \\ [\![q_0+1]\!] \oplus -f_{v,u} ([\![q_2+1, q_3, \dots, q_r]\!]) & \text{if } v \nmid q_0 \text{ and } q_1 \\ [\![q_0+1]\!] \oplus f_{v,u} ([\![-2, q_2+1, q_3, \dots, q_r]\!]) & \text{if } v \nmid q_0, q_1 = 2 \\ [\![q_0]\!] \oplus f_{v,u} ([\![q_1, q_2, \dots, q_r]\!]) & \text{if } v \nmid q_0, q_1 = 2 \\ [\![q_0]\!] \oplus f_{v,u} ([\![q_1, q_2, \dots, q_r]\!]) & \text{otherwise.} \end{cases} \end{aligned} $ $ \end{aligned} \\ \text{Define } g_{u,v} : A_2 \to A_1 \text{ recursively by} \\ g_{u,v} ([\![q_0, q_1, q_2, \dots, q_r]\!]) & \text{if either } q_1 < 0 \text{ and } u \neq 2, \text{ or } q_1 < -2 \text{ and } u = 2 \\ [\![q_0-1, 2]\!] \oplus g_{v,u} ([\![q_2-1, q_3, \dots, q_r]\!]) & \text{if } q_1 = -2, r > 1, \text{ and } u = 2, \end{cases} \end{aligned}$		
$ \begin{array}{l} \text{Define } f_{u,v}: A_1 \to A_2 \text{ recursively by} \\ \\ f_{u,v}(\llbracket q_0, q_1, q_2, \ldots, q_r \rrbracket) & \text{if } r = 0, \\ \llbracket q_0 \rrbracket & \text{if } r = 0, \\ \llbracket q_0 \rrbracket \amalg \oplus -f_{v,u}(\llbracket q_2 + 1, q_3, \ldots, q_r \rrbracket) & \text{if } v \nmid q_0 \text{ and } q_1 \\ \llbracket q_0 + 1 \rrbracket \oplus f_{v,u}(\llbracket -2, q_2 + 1, q_3, \ldots, q_r \rrbracket) & \text{if } v \nmid q_0, q_1 = 2 \\ \llbracket q_0 + 1, -2 \rrbracket & \text{if } v \nmid q_0, q_1 = 2 \\ \llbracket q_0 \rrbracket \oplus f_{v,u}(\llbracket q_1, q_2, \ldots, q_r \rrbracket) & \text{otherwise.} \end{array} $ $ \begin{array}{l} \text{Define } g_{u,v}: A_2 \to A_1 \text{ recursively by} \\ \\ g_{u,v}(\llbracket q_0, q_1, q_2, \ldots, q_r \rrbracket) & \text{if either } q_1 < 0 \text{ and } u \neq 2, \text{ or } q_1 < -2 \text{ and } u = 2 \\ \llbracket q_0 - 1, 2 \rrbracket \oplus g_{v,u}(\llbracket q_2 - 1, q_3, \ldots, q_r \rrbracket) & \text{if } q_1 = -2, r > 1, \text{ and } u = 2, \end{array} $		-
$ \begin{split} f_{u,v}(\llbracket q_0, q_1, q_2, \dots, q_r \rrbracket) & \text{ if } r = 0, \\ \llbracket q_0 \rrbracket & \amalg f r = 0, \\ \llbracket q_0 + 1 \rrbracket \oplus -f_{v,u}(\llbracket q_2 + 1, q_3, \dots, q_r \rrbracket) & \text{ if } v \nmid q_0 \text{ and } q_1 \\ \llbracket q_0 + 1 \rrbracket \oplus f_{v,u}(\llbracket -2, q_2 + 1, q_3, \dots, q_r \rrbracket) & \text{ if } v \nmid q_0, q_1 = 2 \\ \llbracket q_0 + 1, -2 \rrbracket & \text{ if } v \nmid q_0, q_1 = 2 \\ \llbracket q_0 \rrbracket \oplus f_{v,u}(\llbracket q_1, q_2, \dots, q_r \rrbracket) & \text{ otherwise.} \end{split} $ Define $g_{u,v} : A_2 \to A_1$ recursively by $ \begin{aligned} g_{u,v}(\llbracket q_0, q_1, q_2, \dots, q_r \rrbracket) & \text{ if either } q_1 < 0 \text{ and } u \neq 2, \text{ or } q_1 < -2 \text{ and } u = 2 \\ \llbracket q_0 - 1, 2 \rrbracket \oplus g_{v,u}(\llbracket q_2 - 1, q_3, \dots, q_r \rrbracket) & \text{ if } q_1 = -2, r > 1, \text{ and } u = 2, \end{aligned} $		
$ = \begin{cases} \llbracket q_0 \rrbracket & \text{if } r = 0, \\ \llbracket q_0 + 1 \rrbracket \oplus -f_{v,u}(\llbracket q_2 + 1, q_3, \dots, q_r \rrbracket) & \text{if } v \nmid q_0 \text{ and } q_1 \\ \llbracket q_0 + 1 \rrbracket \oplus f_{v,u}(\llbracket -2, q_2 + 1, q_3, \dots, q_r \rrbracket) & \text{if } v \nmid q_0, q_1 = 2 \\ \llbracket q_0 + 1, -2 \rrbracket & \text{if } v \nmid q_0, q_1 = 2 \\ \llbracket q_0 \rrbracket \oplus f_{v,u}(\llbracket q_1, q_2, \dots, q_r \rrbracket) & \text{otherwise.} \end{cases} $ $ Define g_{u,v} : A_2 \to A_1 \text{ recursively by} $ $ \begin{aligned} g_{u,v}(\llbracket q_0, q_1, q_2, \dots, q_r \rrbracket) & \\ & [q_0 \rrbracket & [q_0 \rrbracket] \\ \llbracket q_0 - 1, 1 \rrbracket \oplus g_{v,u}(-\llbracket q_1 + 1, q_2, \dots, q_r \rrbracket) & \\ & \text{if either } q_1 < 0 \text{ and } u \neq 2, \text{ or } q_1 < -2 \text{ and } u = 2 \\ \llbracket q_0 - 1, 2 \rrbracket \oplus g_{v,u}(\llbracket q_2 - 1, q_3, \dots, q_r \rrbracket) & \\ & \text{if } q_1 = -2, r > 1, \text{ and } u = 2, \end{aligned} $	Define $f_{u,v}: A_1 \rightarrow A_2$ recursively by	
$ = \begin{cases} \llbracket q_0 \rrbracket & \text{if } r = 0, \\ \llbracket q_0 + 1 \rrbracket \oplus -f_{v,u}(\llbracket q_2 + 1, q_3, \dots, q_r \rrbracket) & \text{if } v \nmid q_0 \text{ and } q_1 \\ \llbracket q_0 + 1 \rrbracket \oplus f_{v,u}(\llbracket -2, q_2 + 1, q_3, \dots, q_r \rrbracket) & \text{if } v \nmid q_0, q_1 = 2 \\ \llbracket q_0 + 1, -2 \rrbracket & \text{if } v \nmid q_0, q_1 = 2 \\ \llbracket q_0 \rrbracket \oplus f_{v,u}(\llbracket q_1, q_2, \dots, q_r \rrbracket) & \text{otherwise.} \end{cases} $ $ Define g_{u,v} : A_2 \to A_1 \text{ recursively by} $ $ \begin{aligned} g_{u,v}(\llbracket q_0, q_1, q_2, \dots, q_r \rrbracket) & \\ & [q_0 \rrbracket & \\ \llbracket q_0 - 1, 1 \rrbracket \oplus g_{v,u}(-\llbracket q_1 + 1, q_2, \dots, q_r \rrbracket) & \\ & \text{if either } q_1 < 0 \text{ and } u \neq 2, \text{ or } q_1 < -2 \text{ and } u = 2 \\ \llbracket q_0 - 1, 2 \rrbracket \oplus g_{v,u}(\llbracket q_2 - 1, q_3, \dots, q_r \rrbracket) & \\ & \text{if } q_1 = -2, r > 1, \text{ and } u = 2, \end{aligned} $	\boldsymbol{f} (Toronor or T)	
$= \begin{cases} g_{u,v}(\llbracket q_0, q_1, q_2, \dots, q_r \rrbracket) \\ \llbracket q_0 \rrbracket \\ \llbracket q_0 - 1, 1 \rrbracket \oplus g_{v,u}(-\llbracket q_1 + 1, q_2, \dots, q_r \rrbracket) \\ & \text{if either } q_1 < 0 \text{ and } u \neq 2, \text{ or } q_1 < -2 \text{ and } u = 2 \\ \llbracket q_0 - 1, 2 \rrbracket \oplus g_{v,u}(\llbracket q_2 - 1, q_3, \dots, q_r \rrbracket) \\ & \text{if } q_1 = -2, r > 1, \text{ and } u = 2, \end{cases}$	$J_{u,v}(\llbracket q_0, q_1, q_2, \ldots, q_r \rrbracket)$	'f O
$= \begin{cases} g_{u,v}(\llbracket q_0, q_1, q_2, \dots, q_r \rrbracket) \\ \llbracket q_0 \rrbracket \\ \llbracket q_0 - 1, 1 \rrbracket \oplus g_{v,u}(-\llbracket q_1 + 1, q_2, \dots, q_r \rrbracket) \\ & \text{if either } q_1 < 0 \text{ and } u \neq 2, \text{ or } q_1 < -2 \text{ and } u = 2 \\ \llbracket q_0 - 1, 2 \rrbracket \oplus g_{v,u}(\llbracket q_2 - 1, q_3, \dots, q_r \rrbracket) \\ & \text{if } q_1 = -2, r > 1, \text{ and } u = 2, \end{cases}$	$\begin{bmatrix} q_0 \end{bmatrix}$	If $r = 0$,
$= \begin{cases} g_{u,v}(\llbracket q_0, q_1, q_2, \dots, q_r \rrbracket) \\ \llbracket q_0 \rrbracket \\ \llbracket q_0 - 1, 1 \rrbracket \oplus g_{v,u}(-\llbracket q_1 + 1, q_2, \dots, q_r \rrbracket) \\ & \text{if either } q_1 < 0 \text{ and } u \neq 2, \text{ or } q_1 < -2 \text{ and } u = 2 \\ \llbracket q_0 - 1, 2 \rrbracket \oplus g_{v,u}(\llbracket q_2 - 1, q_3, \dots, q_r \rrbracket) \\ & \text{if } q_1 = -2, r > 1, \text{ and } u = 2, \end{cases}$	$= \int [[q_0 + 1]] \oplus -J_{v,u}([[q_2 + 1, q_3, \dots, q_r]])$ $= \int [[q_0 + 1]] \oplus f_{v,u}([[q_2 + 1, q_3, \dots, q_r]])$	if $v \nmid q_0$ and q_1
$= \begin{cases} g_{u,v}(\llbracket q_0, q_1, q_2, \dots, q_r \rrbracket) \\ \llbracket q_0 \rrbracket \\ \llbracket q_0 - 1, 1 \rrbracket \oplus g_{v,u}(-\llbracket q_1 + 1, q_2, \dots, q_r \rrbracket) \\ & \text{if either } q_1 < 0 \text{ and } u \neq 2, \text{ or } q_1 < -2 \text{ and } u = 2 \\ \llbracket q_0 - 1, 2 \rrbracket \oplus g_{v,u}(\llbracket q_2 - 1, q_3, \dots, q_r \rrbracket) \\ & \text{if } q_1 = -2, r > 1, \text{ and } u = 2, \end{cases}$	$- \int [[q_0 + 1]] \oplus J_{v,u}([-2, q_2 + 1, q_3, \dots, q_r]])$ $[[q_0 + 1] - 2]]$	if $v \nmid q_0, q_1 = 2$ if $v \nmid q_0, q_1 = 2$
$= \begin{cases} g_{u,v}(\llbracket q_0, q_1, q_2, \dots, q_r \rrbracket) \\ \llbracket q_0 \rrbracket \\ \llbracket q_0 - 1, 1 \rrbracket \oplus g_{v,u}(-\llbracket q_1 + 1, q_2, \dots, q_r \rrbracket) \\ & \text{if either } q_1 < 0 \text{ and } u \neq 2, \text{ or } q_1 < -2 \text{ and } u = 2 \\ \llbracket q_0 - 1, 2 \rrbracket \oplus g_{v,u}(\llbracket q_2 - 1, q_3, \dots, q_r \rrbracket) \\ & \text{if } q_1 = -2, r > 1, \text{ and } u = 2, \end{cases}$	$\llbracket q_0 \rrbracket \oplus f_n \lrcorner (\llbracket q_1, q_2, \ldots, q_n \rrbracket)$	otherwise. -2
$= \begin{cases} g_{u,v}(\llbracket q_0, q_1, q_2, \dots, q_r \rrbracket) \\ \llbracket q_0 \rrbracket \\ \llbracket q_0 - 1, 1 \rrbracket \oplus g_{v,u}(-\llbracket q_1 + 1, q_2, \dots, q_r \rrbracket) \\ & \text{if either } q_1 < 0 \text{ and } u \neq 2, \text{ or } q_1 < -2 \text{ and } u = 2 \\ \llbracket q_0 - 1, 2 \rrbracket \oplus g_{v,u}(\llbracket q_2 - 1, q_3, \dots, q_r \rrbracket) \\ & \text{if } q_1 = -2, r > 1, \text{ and } u = 2, \end{cases}$	Define $a \rightarrow A_1$ recursively by	
$= \begin{cases} g_{u,v}(\llbracket q_0, q_1, q_2, \dots, q_r \rrbracket) \\ \llbracket q_0 \rrbracket \\ \llbracket q_0 - 1, 1 \rrbracket \oplus g_{v,u}(-\llbracket q_1 + 1, q_2, \dots, q_r \rrbracket) \\ & \text{if either } q_1 < 0 \text{ and } u \neq 2, \text{ or } q_1 < -2 \text{ and } u = 2 \\ \llbracket q_0 - 1, 2 \rrbracket \oplus g_{v,u}(\llbracket q_2 - 1, q_3, \dots, q_r \rrbracket) \\ & \text{if } q_1 = -2, r > 1, \text{ and } u = 2, \end{cases}$	Define $g_{u,v}$. 112 7 111 recursively by	
$= \begin{cases} \llbracket q_0 \rrbracket \\ \llbracket q_0 - 1, 1 \rrbracket \oplus g_{v,u}(-\llbracket q_1 + 1, q_2, \dots, q_r \rrbracket) \\ \text{ if either } q_1 < 0 \text{ and } u \neq 2, \text{ or } q_1 < -2 \text{ and } u = 2 \\ \llbracket q_0 - 1, 2 \rrbracket \oplus g_{v,u}(\llbracket q_2 - 1, q_3, \dots, q_r \rrbracket) \\ \text{ if } q_1 = -2, r > 1, \text{ and } u = 2, \end{cases}$		
$= \begin{cases} \llbracket q_0 - 1, 1 \rrbracket \oplus g_{v,u}(-\llbracket q_1 + 1, q_2, \dots, q_r \rrbracket) \\ \text{if either } q_1 < 0 \text{ and } u \neq 2, \text{ or } q_1 < -2 \text{ and } u = 2 \\ \llbracket q_0 - 1, 2 \rrbracket \oplus g_{v,u}(\llbracket q_2 - 1, q_3, \dots, q_r \rrbracket) \\ \text{if } q_1 = -2, r > 1, \text{ and } u = 2, \end{cases}$		
$= \begin{cases} \llbracket q_0 - 1, 2 \rrbracket \oplus g_{v,u}(\llbracket q_2 - 1, q_3, \dots, q_r \rrbracket) \\ \text{if } q_1 = -2, r > 1, \text{ and } u = 2, \end{cases}$		
$= \begin{cases} \llbracket q_0 - 1, 2 \rrbracket \oplus g_{v,u}(\llbracket q_2 - 1, q_3, \dots, q_r \rrbracket) \\ \text{if } q_1 = -2, r > 1, \text{ and } u = 2, \end{cases}$	if either $q_1 < 0$ and $u \neq 2$, or q_1	< -2 and $u = 2$
$ \begin{bmatrix} -\\ if \ q_1 = -2, \ r > 1, \text{ and } u = 2, \\ \llbracket q_0 - 1, 2 \rrbracket \\ if \ q_1 = -2, \ r = 1, \text{ and } u = 2, \\ \llbracket q_0 \rrbracket \oplus g_{v,u}(\llbracket q_1, q_2, \dots, q_r \rrbracket) $	$[[q_0 - 1, 2]] \oplus g_{v,u}([[q_2 - 1, q_3, \dots, q_r]])$	
$ \begin{split} \llbracket q_0 - 1, 2 \rrbracket \\ & \text{ if } q_1 = -2, r = 1, \text{ and } u = 2, \\ \llbracket q_0 \rrbracket \oplus g_{v,u} (\llbracket q_1, q_2, \dots, q_r \rrbracket) \end{split} $	if $q_1 = -2$, $r > 1$, and $u = 2$,	
$ \text{if } q_1 = -2, r = 1, \text{and} u = 2, \\ \llbracket q_0 \rrbracket \oplus g_{v,u}(\llbracket q_1, q_2, \dots, q_r \rrbracket) $	$\llbracket q_0-1,2\rrbracket$	
$\llbracket q_0 rbracket \oplus g_{v,u}(\llbracket q_1,q_2,\ldots,q_r rbracket)$	if $q_1 = -2$, $r = 1$, and $u = 2$,	
	$\llbracket q_0 rbracket \oplus g_{v,u}(\llbracket q_1,q_2,\ldots,q_r rbracket)$	



Results

 $, q_2, \ldots, q_r].$ Let

 $[p_0, p_1, p_2, \ldots, p_n]$ $[p_1, p_2, \ldots, p_n]$ e.

 $\langle i \langle r \rangle$,

 \rightarrow , and

ation of x. We say rty if $v|q_i$ when i is

 $y_1 = 1$, 2, and r > 1, 2, and r = 1,

if
$$r = 0$$
,

n Ζ,

otherwise.

isfies the (u, v)-divisibility property, then

Proposition 2 Let $[\![q_0, q_1, q_2, \ldots, q_r]\!] \in A_2$. If $[\![q_0, q_1, q_2, \ldots, q_r]\!]$ sat $f_{u,v}(g_{u',v'}(\llbracket q_0, q_1, q_2, \dots, q_r \rrbracket)) = \llbracket q_0, q_1, q_2, \dots, q_r \rrbracket.$ for any positive integers $u', v' \ge 2$ with $u'v' \ne 4$. **Proposition 3** For a matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathscr{G}_{u,v}$, $(f_{u,v} \circ C)(c/a)$ satisfies the (v, u)-divisibility property if and only if $(f_{u,v} \circ C)(b/d)$ satisfies the (u, v)-divisibility property. Proposition 3 allows us to state our main theorem below in terms **Theorem 4** For integers $u, v \ge 2$, with $uv \ne 4$, and a matrix M = $\in \mathscr{G}_{u,v}$, $M \in G_{u,v}$ if and only if $(f_{u,v} \circ C) (b/d)$ satisfies the (u, v)-divisibility property. A careful reading of Theorem 4 shows that our method allows one to determine the exponents in the alternating product representation of M, should it be the case that $M \in G_{u,v}$.

of b/d, ignoring c/a altogether.

An example

 $(f_{2,3} \circ C) \left(\frac{12975}{1351}\right) = f_{2,3}(\llbracket 9, 1, 1, 1, 1, 9, 2, 2, 5 \rrbracket)$ $= \llbracket 9 \rrbracket \oplus f_{3,2}(\llbracket 1, 1, 1, 1, 9, 2, 2, 5 \rrbracket)$

$$= [9] \oplus ([2]] \oplus -f_{2,3}([2,1,9,2,2,5]]))$$

$$= [9] \oplus ([2]] \oplus -([3]] \oplus -f_{3,2}([10,2,2]))$$

$$= [9] \oplus ([2]] \oplus -([3]] \oplus -([10]] \oplus f_{3,2}([10,2,2]))$$

$$= [9] \oplus ([2] \oplus -([3] \oplus -$$
$$= [9] \oplus ([2] \oplus -([3] \oplus -$$

$$f_{22} (\llbracket -2 6 \rrbracket))))$$

$$= [9] \oplus ([2] \oplus -([3] \oplus -([f_{2,3}([6])))))$$

$$= \llbracket 9 \rrbracket \oplus (\llbracket 2 \rrbracket \oplus -(\llbracket 3 \rrbracket \oplus -(\llbracket 6 \rrbracket))))$$

$$= [[9, 2, -3, 10, 3, -2, 6]],$$

which does satisfy the (2,3)-divisibility property, as desired, and encodes the exponents in the product representation of M = $\begin{bmatrix} 2401 & 12975 \\ 250 & 1351 \end{bmatrix} = R_3^3 L_2 R_3^{-1} L_2^5 R_3 L_2^{-1} R_3^2.$

References

[1] Henri-Alex Esbelin and Marin Gutan. "On the membership problem for some subgroups of $SL_2(\mathbb{Z})$ ". In: Ann. Math. Qué. 43.2 (2019), pp. 233–247. [2] Sandie Han et al. "Subgroups of $SL_2(\mathbb{Z})$ characterized by certain continued fraction representations". In: Proc. Amer. Math. Soc. 148.9 (2020), pp. 3775-3786.

 $f_{3,2}(\llbracket 10, 2, 2, 5 \rrbracket)))$ $(\llbracket 10 \rrbracket \oplus f_{2,3}(\llbracket 2, 2, 5 \rrbracket)))$ $\cdot (\llbracket 10 \rrbracket \oplus (\llbracket 3 \rrbracket \oplus$ $(\llbracket 10 \rrbracket \oplus (\llbracket 3, -2 \rrbracket \oplus$

 $(\llbracket 10 \rrbracket \oplus (\llbracket 3, -2 \rrbracket \oplus$