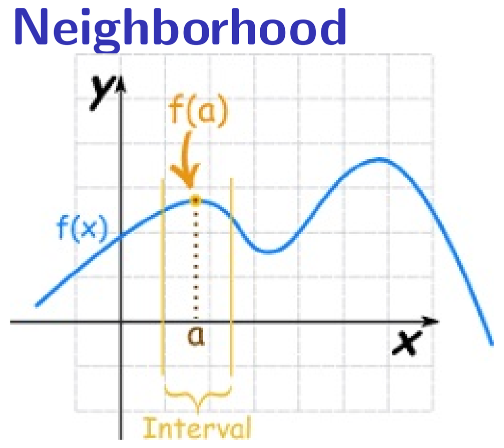
The theory of compact spaces

Hans Schoutens
Mathematics

Abstract

While topological spaces can be very 'wild', compactness introduces enough amount of finiteness to guarantee a much 'tamer' behavior. We would therefore expect that the class of all compact spaces would have a nicer model-theoretic description, and this is indeed the case. It turns out that we can recover the topology on each space just from the properties of the entire structure/category (=the class of all compact spaces and the continuous maps between them), as I will explain in this poster. In particular, each structure/category contains a model of Peano Arithmetic, that is to say, it has its own version of the natural numbers.

Neighborhood



Q What does it mean that a function f has a local maximum at a point a?
A That in a *neighborhood* of a, the function f has lower values than at a.
Q But what does *in a neighborhood* mean?
A That depends on the *topology* you use!

Topology

Definition (Topology)

A *topology* on a set X consists of a collection of *open* sets, including X and $\emptyset$. The collection must be closed under finite intersections and arbitrary unions. The complement of an open set is called *closed*.

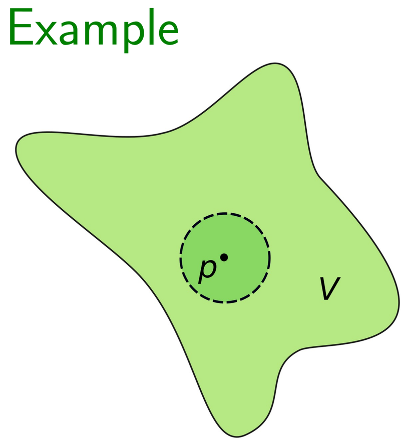
Definition

A *neighborhood* N of a point $a \in X$ is a subset containing an open U containing a.

Definition (Spaces)

A space is *Hausdorff*, if any two points have disjoint neighborhoods. Any set endowed with a Hausdorff topology is called a *space*.

Example



- On the real line, any interval without endpoints is open
- A neighborhood of a point P in the Cartesian plane is any set containing an open disk centered at P.
- Any set becomes a *discrete space*, by declaring every subset open.
- For the Zariski topology on a variety, the closed subsets are just the subvarieties. However, this is in general a non-Hausdorff topology.

Compact spaces

Definition

A collection of opens is said to *cover* a space X, if their union is equal to X. We say that X is *compact*, if in any covering already finitely many can cover the space.

Example

- The closed unit interval is compact.
- The real line is not compact: the open intervals $(n, n+2)$, for $n \in \mathbb{Z}$ cover the line, but no finite sub-collection does.
- Any closed disk in the plane is compact.
- Any variety is compact; any closed manifold is compact.
- A discrete space is compact if and only if it is finite; in particular, the one-point space $\{*\}$ is compact.

The category of compact spaces

Definition (Continuity)

A map $f\colon X \to Y$ between spaces is called *continuous*, if the pre-image $f^{-1}(U)$ of any open $U \subseteq Y$ is an open in X.

Example

- Any map between varieties given by polynomial equations is continuous.
- The characteristic function of a proper subset of the real line is not continuous.

Definition (Category)

A *category* is a collection of objects and morphisms between them. In the category of topological spaces, the objects are the spaces and the morphisms are the continuous maps. We are interested in the subcategory of all *compact spaces*.

Example

- For any space, there is a unique morphism from the empty space into it; we say that the empty space is an *initial* object.
- Any space admits a unique morphism onto the one-point space $\{*\}$; we say that $\{*\}$ is a *terminal* object.
- A category admits *products*, if there are morphisms $X_1 \times X_2 \to X_i$, such that any pair of morphisms $Z \to X_i$ factor through a unique morphism $Z \to X_1 \times X_2$; if the arrows go the other way, we speak of a *co-product* and denote it $X_1 \sqcup X_2$.

The theory of compact spaces

Our goal is to generalize the category of compact spaces. The key fact is that the topology on any of its objects can be recovered from the morphisms:

Fact

If $X \to Y$ is a morphism (=continuous map) of compact spaces, then the image of f is closed.

Definition (The first-order structure of spaces)

- Consider a structure $\mathcal{T}$ in which the elements are (formal) arrows $X \to Y$; the symbols X and Y are called respectively the *domain* and *co-domain* of the arrow. Included are the 'empty' arrow $\emptyset$ and the 'terminal arrow' $\{*\} \to \{*\}$. We also assume that we can 'compose' compatible arrows $X \to Y$ and $Y \to Z$, and that there are identity arrows $1_X\colon X \to X$. We assume that we can take products $\times$ and co-products $\sqcup$ of arrows, and that every arrow has an 'image'.
- Any arrow of the form $X \to \{*\}$ will be called a *space*, and we will just denote it by X.
- We define a *point* of a space X as an arrow $\{*\} \to X$ and we call the collection of all points $|X|$ the *underlying set of points*. Any arrow $X \to Y$ induces a map $|X| \to |Y|$ by composition, and we require that two arrows inducing the same map must be equal.
- We define a 'topology' on $|X|$ by taking for closed sets the images of arrows $Y \to X$ (and for opens the complements of images). We require that any two points can be separated by opens.

Discrete spaces

Definition

Let X be a space. We call a point $a \in |X|$ *isolated*, if $\{a\}$ is open. We call X *discrete*, if every point is isolated.

SLOGAN: Discrete spaces behave like finite spaces!

Theorem (Discrete Pigeonhole Principle)

Any definable injective self-map on a discrete space is surjective, and conversely.

Compactness axiom: any definable open covering has a discrete subcovering.

Limits

Example

There is up to isomorphism exactly one countable compact topological space with a single non-isolated point, namely

$$I := \{0, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4} \dots\} \cong [0, \omega] \tag{1}$$

where the latter is the one-point compactification of $\mathbb{N}$. Moreover, any space with a single non-isolated point contains I.

We require that an I satisfying the latter condition exists in an arbitrary structure $\mathcal{T}$. Moreover, we stipulate that a subset $V \subseteq |X|$ is closed if and only if any arrow $I \to X$ mapping the isolated points of I inside V should also map the unique non-isolated point in V.

The model of PA

Let us say that $A \leq B$, for A and B discrete spaces if there is an arrow $A \to B$ such that the underlying map is injective. In particular, if $a \in |A|$ is some point, then we can remove this point and the resulting discrete space is required to be strictly smaller. Let PA be the collection of all isomorphism classes of discrete sets. Addition and multiplication of elements of PA are given by the co-product and product respectively

$$A + B := A \sqcup B \qquad \text{and} \qquad A \cdot B := A \times B.$$

Definable completeness any non-empty definable collection of discrete spaces has a least element.

Main Theorem

The set PA is a model of Peano Arithmetic, that is to say, is a (non-standard) version of the natural numbers.

Main Theorem

Any (Cartesian) category of compact topological spaces is a structure satisfying the above axioms; these are the only structures for which $PA = \mathbb{N}$.

Puzzling Consequence

In the category of compact topological spaces, any subset is definable!

A question for topologists: Does compactness define cardinals?

Given a space X, let ιX be obtained from X by identifying all its non-isolated points. Let us say that a non-discrete space X is $\mathcal{T}$*-countable* if $\iota Y \cong I$ (see (1)), for any non-discrete subspace Y of X.

Conjecture

A compact topological space is $\mathcal{T}$-countable if and only if it is countable.

Justification.

The converse implication holds, and any attempt to construct a counterexample to the other direction has so far failed! □