A conjugate directions-type procedure for quadratic multiobjective optimization

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Abstract

We propose an extension of the real-valued conjugate directions method for unconstrained quadratic multiobjective problems. As in the single-valued counterpart, the procedure requires a set of directions that are simultaneously conjugate with respect to the positive definite matrices of all quadratic objective components. Likewise, the multicriteria version computes the steplength by means of the unconstrained minimization of a single-variable strongly convex function at each iteration. When it is implemented with a weakly-increasing (strongly-increasing) auxiliary function, the scheme produces weak Pareto (Pareto) optima in finitely many iterations.

The Quadratic Single Objective Problem

Let $Q \in \mathbb{R}^{n\times n}$ be positive definite and $q \in \mathbb{R}^n$. Consider the quadratic function $f: \mathbb{R}^n \to \mathbb{R}$ by

$$f(x) = \frac{1}{2}x^TQx + q^Tx.$$

In this case the minimization problem is $\min_{x\in\mathbb{R}^n} f(x)$. Since f is strongly convex, the minimizer is unique, say x^*.

For this problem, the following method is a classical one. It guarantees to find the minimizer in at most n steps.

The Scalar Conjugate Directions Method

Goal: To solve $\min_{x\in\mathbb{R}^n} f(x)$, that is, to find $x^* \in \mathbb{R}^n$ such that $f(x^*) \le f(x)\ \forall x \in \mathbb{R}^n$.

- Step 1: Choose Q-conjugate directions $d^0, d^1, \dots, d^{n-1}$ in $\mathbb{R}^n$, i.e.

$$\langle d^i, d^j\rangle_Q = (d^i)^TQd^j = 0 \quad \forall i \ne j.$$

- Step 2: Choose *any* $x^0 \in \mathbb{R}^n$.
- Step 3: For $k = 0, 1, \dots, n-1$, compute the steplenght,

$$t_k = \arg\min_{t\in\mathbb{R}} f(x^k + td^k),$$

and take

$$x^{k+1} = x^k + t_kd^k.$$

The minimizer is $x^* = x^n$.

The Quadratic Multiobjective Problem

Let us consider the extension of the quadratic single objective problem to the following one.

Let $Q_i \in \mathbb{R}^{n\times n}$ be positive definite and $q^i \in \mathbb{R}^n$ for $i = 1, \dots, m$. Consider the quadratic function $f = (f_1, \dots, f_m)\colon \mathbb{R}^n \to \mathbb{R}^m$ given by

$$f_i(x) = \frac{1}{2}x^TQ_ix + (q^i)^Tx,$$

for $i = 1, \dots, m$. Our problem is $\min_{x\in\mathbb{R}^n} f(x)$. The goal is to seek weak Pareto or Pareto optimal points.

Pareto and Weak Pareto optimality

For $u = (u_1, \dots, u_m)$ and $v = (v_1, \dots, v_m)$ in $\mathbb{R}^m$, we have

$$\begin{cases} u \le v & \text{if } u_i \le v_i \text{ for } i = 1, \dots, m, \\ u < v & \text{if } u_i < v_i \text{ for } i = 1, \dots, m. \end{cases}$$

A point $x^* \in \mathbb{R}^n$ is called a

$$\begin{cases} \textit{weak Pareto optimal solution} & \text{if } \nexists\ x \in \mathbb{R}^n \text{ with } f(x) < f(x^*), \\ \textit{Pareto optimal solution} & \text{if } \nexists\ x \in \mathbb{R}^n \text{ with } f(x) \le f(x^*) \text{ and } f(x) \ne f(x^*). \end{cases}$$

- Every Pareto optimal solution is weak Pareto.

Monotonic Functions

A mapping $\Phi : \mathbb{R}^m \to \mathbb{R}$ is

$$\begin{cases} \text{weakly increasing} & \text{if } \Phi(u) < \Phi(v) \text{ whenever } u < v, \\ \text{strongly increasing} & \text{if } \Phi(u) < \Phi(v) \text{ whenever } u \le v \text{ and } u \ne v. \end{cases}$$

- Any strongly increasing function is weakly increasing.

Scalarization

Proposition: Let $g : \mathbb{R}^n \to \mathbb{R}^m$, $\Phi : \mathbb{R}^m \to \mathbb{R}$, and $\bar{x} \in \arg\min_{x\in M\subset\mathbb{R}^n} \Phi\big(g(x)\big)$.

1. If Φ is weakly increasing, then $\bar{x}$ is a weak Pareto optimal solution for $\min_{x\in M\subset\mathbb{R}^n} g(x)$.
2. If Φ is strongly increasing, then $\bar{x}$ is a Pareto optimal solution for $\min_{x\in M\subset\mathbb{R}^n} g(x)$.

General Assumption

- Assumption $\mathcal{E}$: There exists a Q_i-conjugate Hamel basis $\{w^0, \dots, w^{n-1}\} \subset \mathbb{R}^n$, where $Q_i \subset \mathbb{R}^{n\times n}$ corresponds to f_i for $i = 1, \dots, m$.

σ^i-relation

Definition

Let $Q_1 = P_1D_1P_1^T, Q_2, \dots, Q_m \in \mathbb{R}^{n\times n}$ be positive definite where P_1 is orthogonal and D_1 is diagonal. For $1 \le i \le m$ and $1 \le j \le n$, let σ^i be a permutation on $\{1, \dots, n\}$ and I_{σ^i} be the matrix whose j-th column is the $\sigma^i(j)$-th column of the identity matrix I of order n. We say that Q_i is σ^i*-related* to Q_1 if $Q_i = (P_1I_{\sigma^i})D_i(P_1I_{\sigma^i})^T$ for $1 \le i \le m$.

Proposition: Let $Q_1 = P_1D_1P_1^T, Q_2, \dots, Q_m \in \mathbb{R}^{n\times n}$ be positive definite where P_1 is orthogonal and D_1 is diagonal. The following conditions are equivalent:

1. Each Q_i is σ^i-related to Q_1 for $i = 1, \dots, m$.
2. $Q_1, \dots, Q_m$ are simultaneously diagonalizable, i.e. $Q_i = P_1\tilde{D}_iP_1^T$ where P_1 is orthogonal and each $\tilde{D}_i$ is diagonal for $i = 1, \dots, m$.
3. $Q_1, \dots, Q_m$ have the same eigenvectors, namely, $P_1e^1, \dots, P_1e^n$.

Furthermore, Assumption $\mathcal{E}$ holds under either one of these conditions.

The Multiobjective Conjugate Directions Method (weak MCDM version)

- Goal: To find a weak Pareto solution $x^* \in \mathbb{R}^n$, that is, to find $x^* \in \mathbb{R}^n$ such that there is no $x \in \mathbb{R}^n$ such that $f(x) < f(x^*)$.
- Step 1: Choose Q_i-conjugate directions $w^0, w^1, \dots, w^{n-1}$ for $i = 1, \dots, m$, that is,

$$\langle w^r, w^s\rangle_{Q_i} = (w^r)^TQ_iw^s = 0 \quad \forall r \ne s,\ \forall i,$$

and take a continuous weakly increasing auxiliary function $\Phi : \mathbb{R}^m \to \mathbb{R}$ such that $\Phi \circ f$ is strongly convex.

- Step 2: Choose *any* $x^0 \in \mathbb{R}^n$.
- Step 3: For $k = 0, \dots, n-1$, compute the steplength,

$$t_k = \arg\min_{t\in\mathbb{R}} \Phi\big(f(x^k + tw^k)\big),$$

and take

$$x^{k+1} = x^k + t_kw^k.$$

Remarks

- $\Phi \circ f\colon \mathbb{R}^n \to \mathbb{R}$ decreases along $\{x^k\}_k$, i.e., we have

$$\Phi\big(f(x^{k+1})\big) \le \Phi\big(f(x^k)\big)$$

for all $k = 0, \dots, n-1$.

- The strong version of *MCDM* is similar to the weak one: it suffices to take a continuous strongly increasing auxiliary function Φ such that $\Phi \circ f$ is strongly convex.
- When $m = 1$, we retrieve the classical method.

Convergence Result

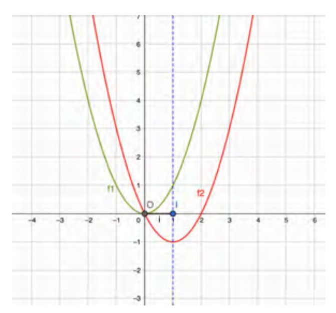
Theorem (Fukuda, Graña Drummond, Masuda - 2021)

Assume that $\{x^k\}_k$ *is generated by MCDM.*

1. *If MCDM is implemented with a weakly increasing auxiliary function* Φ, *then* x^n *is a weak Pareto optimal solution for problem* $\min_{x\in\mathbb{R}^n} f(x)$.
2. *If MCDM is implemented with a strongly increasing auxiliary function* Φ, *then* x^n *is a Pareto optimal solution for problem* $\min_{x\in\mathbb{R}^n} f(x)$.

An *ad hoc* Example

By varying a parameter in the auxiliary function, *MCDM* furnishes the whole Pareto optimal set and consequently the whole Pareto frontier. Let $f\colon \mathbb{R} \to \mathbb{R}^2$ defined by $f(x) = (x^2, x^2 - 2x)$.



The set of Pareto optima is $[0, 1]$. We consider the family of auxiliary functions $\{\Phi_\omega\colon \mathbb{R}^2 \to \mathbb{R}\}_{\omega\in[0,2]}$ defined by $\Phi_\omega(u) = \max_{i=1,2}\{(u - \omega e^1)_i\}$. The function Φ_ω is continuous weakly increasing, and $\Phi_\omega \circ f\colon \mathbb{R} \to \mathbb{R}$ is strongly convex. We apply *MCDM* with $\{w_0\} = \{1\}$ as the basis for $\mathbb{R}$, Φ_ω for $\omega \in [0, 2]$, and $x_0 \in \mathbb{R}$. We obtain a family $\{x^*_\omega\}_{\omega\in[0,2]}$ of Pareto optima such that $\{x^*_\omega\}_{\omega\in[0,2]} = [0, 1]$.

Final remarks

We propose a conjugate directions-type method for unconstrained quadratic multiobjective problems. Essentially, the strategy consists of substituting the unconstrained multicriteria problem by a finite sequence of single-variable unconstrained scalar-valued convex optimization problems, all with a single optimal solution. Depending on the chosen auxiliary function, the procedure yields a weak Pareto or a Pareto optimum.

Our example suggests that it may be worth to investigate which classes of quadratic multiobjective problems are such that the scheme produces all optima by varying a parameter on the auxiliary function. The fact that *MCDP* furnishes an unconstrained minimizer of $\Phi \circ f$ (an apparent limitation) may be a good starting point for characterizing the classes of problems whose efficient frontier can be entirely computed. Another possible research direction is to explore applications of the scheme to multiobjective problems that satisfy weaker conditions.

References

[1] Ellen H. Fukuda, L. M. Graña Drummond, and Ariane M. Masuda.
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