Simulating the Motion of a Rotating Asymmetric Object

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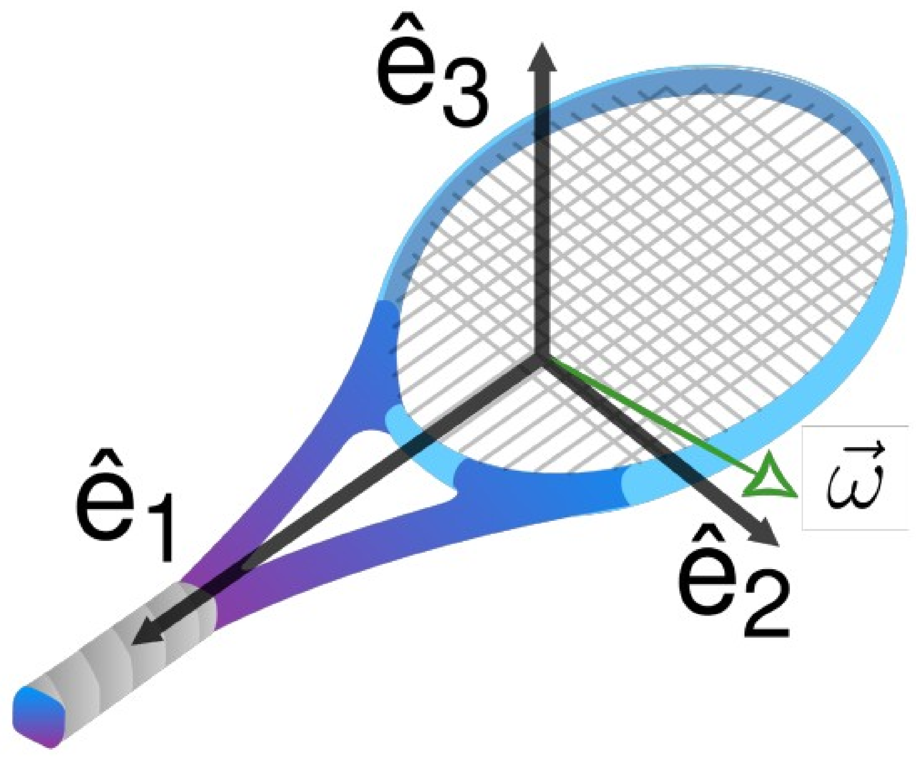
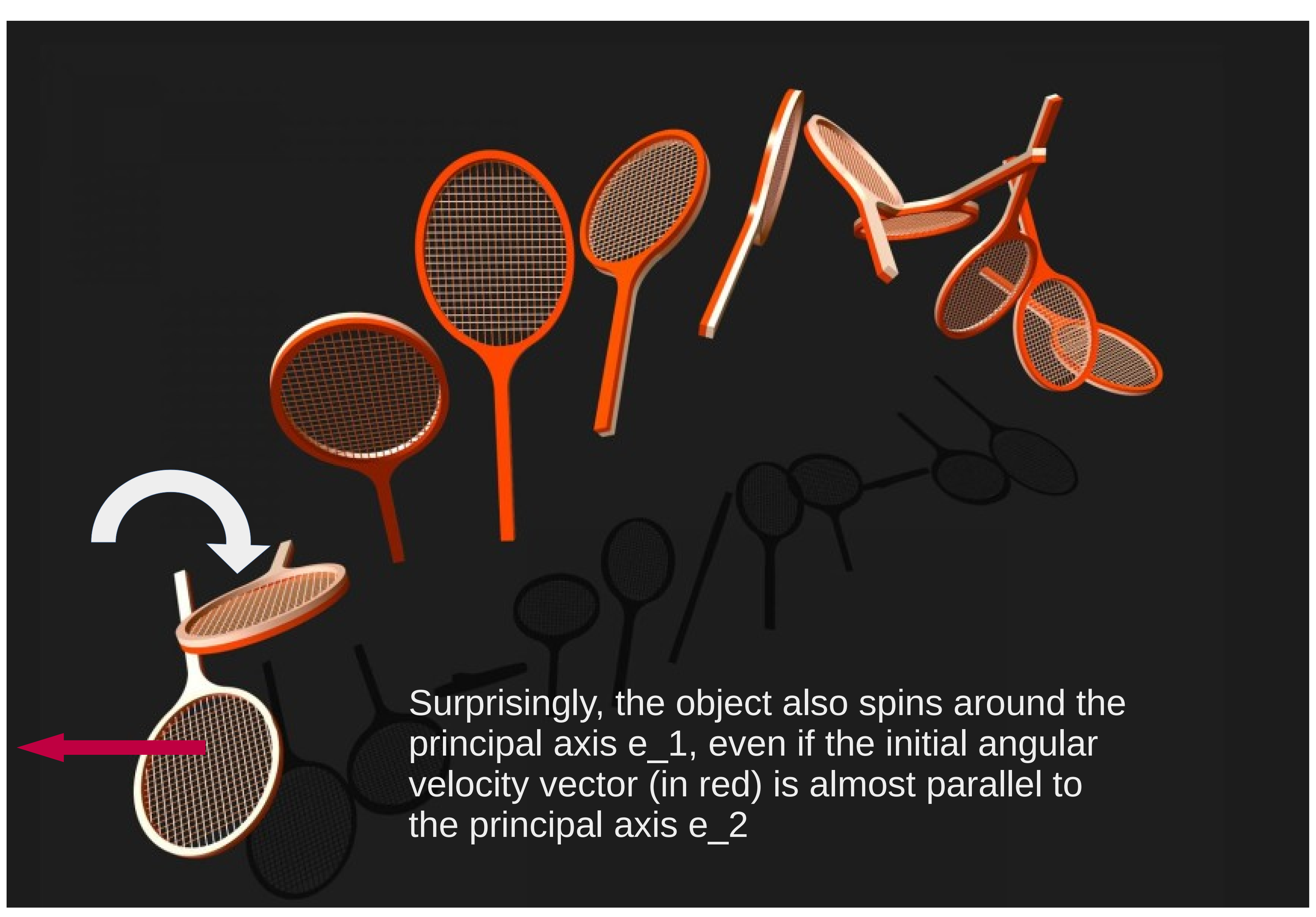
Abstract

The rotational motion of an asymmetric object that is not subject to an external torque shows interesting features when the object has an initial angular velocity that is almost parallel to the principal axis that corresponds to the intermediate moment of inertia. In that case, the frame of reference formed by the principal axes flips periodically in the laboratory frame in a surprising way. This behavior is sometimes referred to as the Dzhanibekov effect. The purpose of this paper is to show how a numerical solution of Euler's equations for a spinning asymmetric object allows one to obtain in a simple way the position of the object's principal axes at each instant in time and to visualize the motion of the rotating object in the laboratory frame. With the procedure outlined in this work it becomes straightforward to write a code that produces an animation of the motion of the rotating object. This exercise could allow students to apply several numerical techniques to a simple but non-trivial problem.

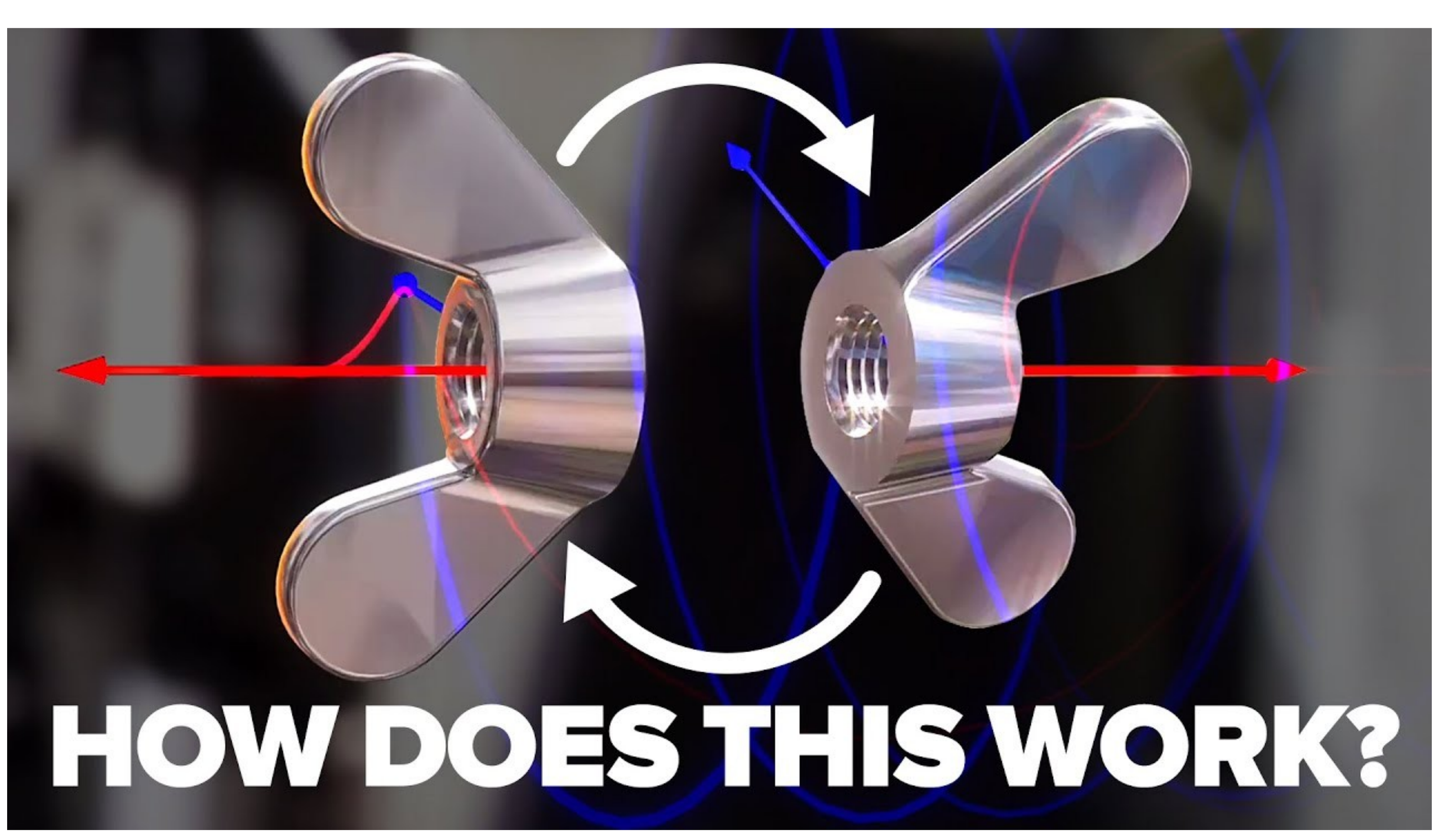


$$I_1 < I_2 < I_3$$

The object flips its axis of rotation when given an initial angular velocity almost parallel to the principal axis corresponding to the intermediate moment of inertia



The effect is particularly striking when observed in a micro gravity environment: see the NASA YouTube video on Dzhanibekhov effect.



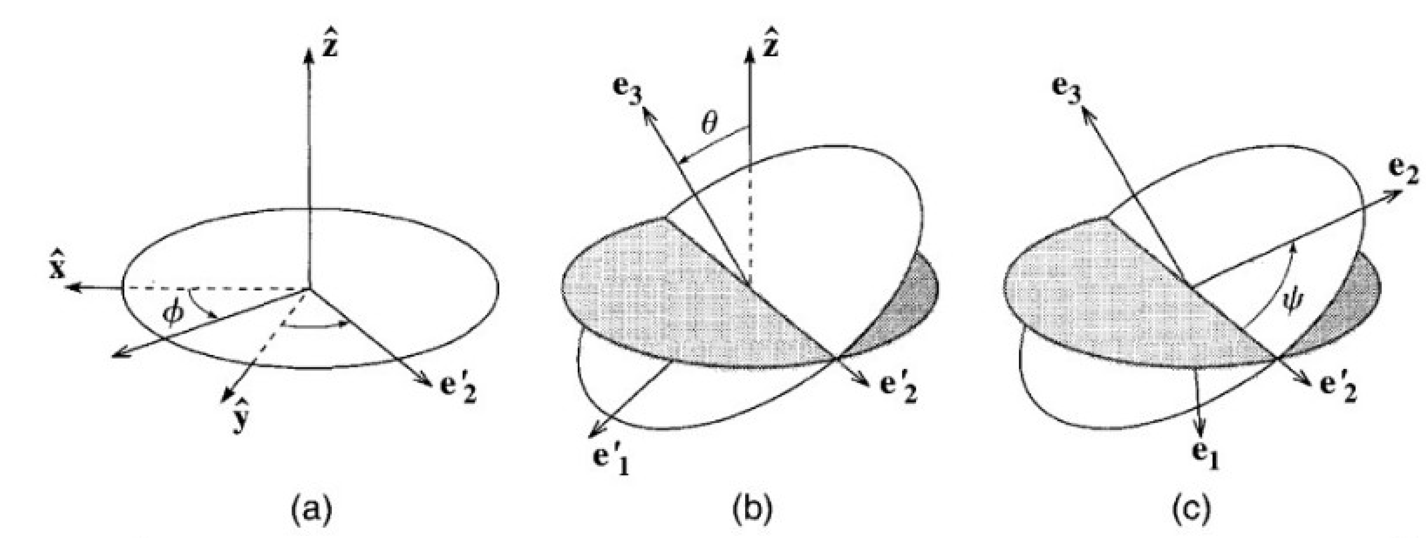
This effect can be explained with the known laws of Newtonian mechanics applied to the rigid body. Although the relevant equations can be solved analytically (Jacobi 1850), here we are interested in solving them numerically and in creating a computer simulation of the object's motion.

Euler's Equations

Assuming that the angular momentum is conserved and the coordinate system coincides with the principal axes of inertia the following equations are satisfied by the components of the angular velocity

$$I_1\dot{\omega}_1 = (I_2 - I_3)\,\omega_2\omega_3$$
$$I_2\dot{\omega}_2 = (I_3 - I_1)\,\omega_1\omega_3$$
$$I_3\dot{\omega}_3 = (I_1 - I_2)\,\omega_1\omega_2$$

These equations can be easily solved numerically. Their solutions can be used to find the Euler angles, that in turns allows one to build the rotation matrix that rotates the components of an arbitrary vector from the body frame to the lab frame. The lab frame is defined as the inertial frame where the conserved angular momentum is aligned to the z axis.



$$\mathbf{R} = \begin{pmatrix} \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi & -\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi & \sin\theta\sin\phi \\ \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi & -\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi & -\sin\theta\cos\phi \\ \sin\theta\sin\psi & \sin\theta\cos\psi & \cos\theta \end{pmatrix}.$$

$$\mathbf{R}\cdot\mathbf{v}_{\text{body frame}} = \mathbf{v}_{\text{lab frame}}$$

Request that the angular momentum is aligned to the z axis

$$\mathbf{R}\cdot\hat{\mathbf{l}} = \hat{\mathbf{k}},$$

This allows one to fix two Euler angles in terms of angular momentum components (i.e. solutions of Euler's equations)

$$\theta = \arccos l_3\,, \qquad \psi = \arctan\frac{l_1}{l_2}\,.$$

The last angle can be fixed by integrating its derivative, that is a function of the solutions of Euler's equations

$$\dot{\phi} = \frac{\omega_1 l_1 + \omega_2 l_2}{l_1^2 + l_2^2}\,,$$

One can then determine the components of the principal axes in the lab frame at each instant in time. This is equivalent to know the orientation of the object at each instant in time.

$$\mathbf{R}\cdot\hat{\mathbf{e}}_1 = \begin{pmatrix} \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi \\ \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi \\ \sin\theta\sin\psi \end{pmatrix} \equiv \hat{\mathbf{p}}\,,$$

$$\mathbf{R}\cdot\hat{\mathbf{e}}_2 = \begin{pmatrix} -\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi \\ -\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi \\ \sin\theta\cos\psi \end{pmatrix} \equiv \hat{\mathbf{q}}\,,$$

$$\mathbf{R}\cdot\hat{\mathbf{e}}_3 = \begin{pmatrix} \sin\theta\sin\phi \\ -\sin\theta\cos\phi \\ \cos\theta \end{pmatrix} \equiv \hat{\mathbf{r}}\,.$$

Simulation

Coded in Python by D. Ramirez, works for arbitrary moments of inertia and initial conditions

$$I_1 = 1 \quad \text{kg m}^2$$
$$I_2 = 2 \quad \text{kg m}^2$$
$$I_3 = 3 \quad \text{kg m}^2$$

$$\omega_1(0) = 10^{-8} \quad \text{rad/s}$$
$$\omega_2(0) = 1 \quad \text{rad/s}$$
$$\omega_3(0) = 10^{-1} \quad \text{rad/s}$$