A Sinusoidal Twist with Exponential Influences

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Abstract

We show that the total distance traveled by an under-damped oscillating spring mass system with sinusoidal displacement results in a nice, closed-form expression.

Motivation

In this study the distance traveled by a mass in an under damped system was simulated with the Maple software. This is an important problem in physical and quantum systems in mathematics and physics. The results touched upon an interaction between calculus and differential equations and provide ample material for student projects and explorations .

Preliminaries

The second-order differential equation that models a spring-mass system is usually encountered in a first course in ordinary differential equations. The equation takes the form

$$mu''(t) + \gamma u'(t) + cu(t) = F(t), \qquad (1)$$

where $u(t)$ is the displacement at time t, m is the mass, γ is the damping constant and c is the spring constant.

The under-damped case with no external force, $F(t) = 0$, is most interesting as a mathematical model for applications in electrical, mechanical and quantum systems. In this case, $F(t) = 0$, the characteristic equation of (1), has the roots $(-\gamma \pm \sqrt{\gamma^2 - 4cm})/2m$ with $\gamma^2 - 4cm < 0$, and its displacement is given by

$$u(t) = e^{-\gamma t/2m}\left(A\cos(\mu t) + B\sin(\mu t)\right)$$

or, equivalently,

$$u(t) = Pe^{-\gamma t/2m}\cos(\mu t - \delta),$$

where $P = \sqrt{A^2 + B^2}$, $\tan(\delta) = B/A$, (the quadrant location of δ is determined based on the signs of A and B), $\gamma > 0$ and $\mu = \sqrt{4cm - \gamma^2}/2m > 0$.

Under-damped Oscillation

The general shape of $u(t)$ is illustrated below.

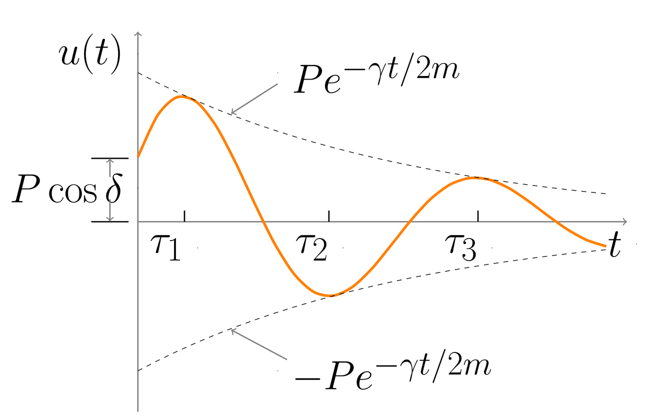

Figure 1: Under-damped oscillation

Parametrization

Vertical displacements at the origin and the extreme values can be parametrized to find the distance traveled by the particle. The distance S traveled by the mass is given by

$$S = \int_0^\infty \sqrt{(0)^2 + (u'(t))^2}\,dt.$$

Computation with Maple

The next step is to use Maple to calculate S when $u(t) = 6e^{-t}\cos\left(t - \frac{\pi}{4}\right)$. Using the Maple code

$u := t \to 6 \cdot e^{-t} \cdot \cos\left(t - \frac{\pi}{4}\right)$:

$S := \int_0^\infty \sqrt{\left(\frac{d}{dt}u(t)\right)^2}\,dt$;

$$S := \frac{3\sqrt{2}(e^\pi + 1)}{e^\pi - 1}.$$

It is an easy exercise to establish that $\frac{3\sqrt{2}(e^\pi+1)}{e^\pi-1} = 3\sqrt{2}\coth(\pi/2)$.

Simulated Example

We began work on the following example. The motion of a spring-mass system is described by the differential equation $u''(t) + 0.5u'(t) + 2u(t) = F(t)$, where $u(t)$ is measured in feet and t in seconds. If $u(0) = 2$ ft and $u'(0) = -1$ ft/s, find the distance traveled by the mass as time approaches infinity. We will experiment with $F(t) = kt^n$, where k and n are nonnegative constants. In particular, we illustrate three cases, $F(t) = 0$, $F(t) = 5$, and $F(t) = t$ respectively with the help of Maple software.

Maple Results

Case I. $u1(t) = -\frac{2\sqrt{31}e^{-\frac{t}{4}}}{31}\sin\left(\frac{\sqrt{31}t}{4}\right) + 2e^{-\frac{t}{4}}\cos\left(\frac{\sqrt{31}t}{4}\right)$ and

$$S1 = \frac{2\left(e^{\frac{\sqrt{31}\pi}{31}} - 1 + 2e^{\frac{\sqrt{31}\arctan\left(\frac{\sqrt{31}}{15}\right)}{31}}\right)}{e^{\frac{\sqrt{31}\pi}{31}} - 1}$$

Case II. $u2(t) = u1(t) + \frac{5}{2}$ and $S1 = S2$.

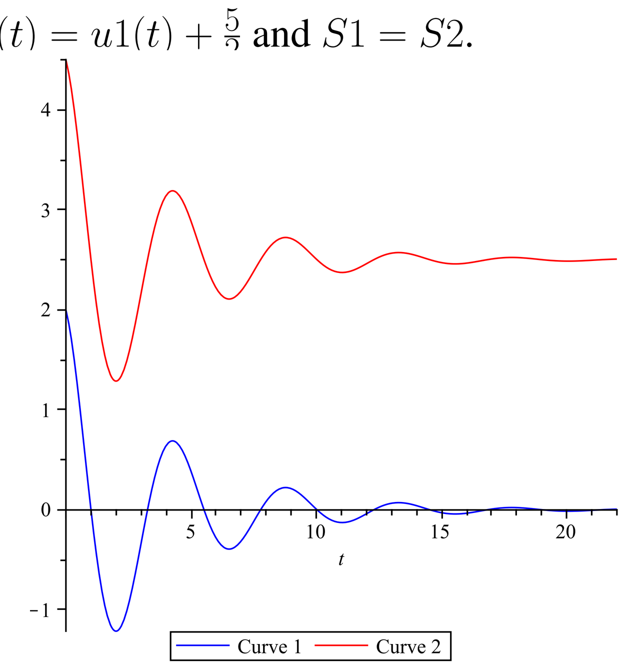

Figure 2: $u1(t)$ and $u2(t)$ verses t.

Case III. $u3(t) = -\frac{\sqrt{31}e^{-\frac{t}{4}}}{8}\sin\left(\frac{\sqrt{31}t}{4}\right) + \frac{17}{8}e^{-\frac{t}{4}}\cos\left(\frac{\sqrt{31}t}{4}\right) - \frac{1}{8} + \frac{t}{2}$.

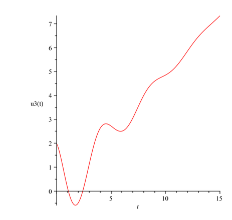

Figure 3: $u3(t)$ verses t.

This shape suggests that the distance traveled by the mass is growing without bound.

Theoretical Computation

A spring mass system is described by the differential equation $u''(t) + u'(t) + u(t) = 0$, where $u(t)$ is measured in feet and t in seconds. If $u(0) = 1$ ft and $u'(0) = -1$ ft/s, we will find the distance traveled by the mass as time approaches infinity by direct calculation. It can be shown that $u(t) = \frac{2}{\sqrt{3}}e^{-\frac{t}{2}}\cos\left(\frac{\sqrt{3}}{2}t + \frac{\pi}{6}\right)$.

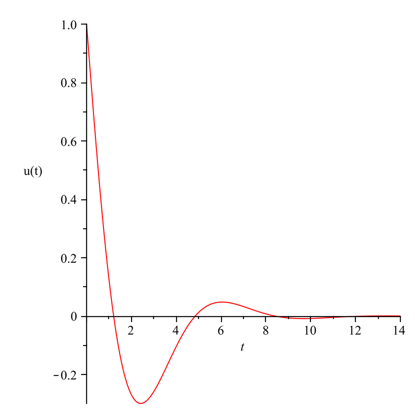

Figure 4: $u(t)$ verses t.

The extreme values of $u(t)$ occur at $\tau_n = \frac{2\pi}{\sqrt{3}}\left(n - \frac{1}{3}\right)$, and its horizontal intercepts occur at $t_m = \frac{2\pi}{3\sqrt{3}}(3m - 2)$, where n and m are positive integers. In the interval $[0, t_1]$ the mass will travel $|u(0)|$, in $[t_1, t_2]$ it will travel $2|u(\tau_1)|$, in $[t_2, t_3]$ it will travel $2|u(\tau_2)|$ and so on. If we denote the distance by S, we get that

$$\begin{aligned} S &= |u(0)| + 2|u(\tau_1)| + 2|u(\tau_2)| + \cdots \\ &= 1 + 2e^{-\frac{2\sqrt{3}\pi}{9}} + 2e^{-\frac{5\sqrt{3}\pi}{9}} + \cdots \\ &= 1 + \frac{2e^{-\frac{2\sqrt{3}\pi}{9}}}{1 - e^{-\frac{\sqrt{3}\pi}{3}}}. \end{aligned}$$

We can rewrite S equivalently as $1 + \left(1 + \coth\left(\frac{\pi}{2\sqrt{3}}\right)\right)e^{-\frac{2\sqrt{3}\pi}{9}}$.

Main Result

Theorem 1. *For an under-damped spring mass system with displacement* $u(t) = Pe^{-\gamma t/2m}\cos(\mu t - \delta)$, *where* $P = \sqrt{A^2 + B^2}$, $\tan(\delta) = B/A$, *(the quadrant location of* δ *is determined based on the signs of* A *and* B*),* $\gamma > 0$, $\mu = \sqrt{4cm - \gamma^2}/2m > 0$ *and* $\tau_1 = \frac{\phi+\delta+k\pi}{\mu}$, *where* $k = \min\{j : \phi + \delta + j\pi > 0, j \in \mathbb{N}\}$, *the distance* S *traveled by the mass before coming to rest is given by the formula*

$$P\left[|\cos(\phi)|\left(1 + \coth(\gamma\pi/4m\mu)\right)e^{-\gamma\tau_1/2m} - |\cos(\delta)|\,\mathrm{sgn}\left(u(0)u(\tau_1)\right)\right].$$

Corollary

Corollary 1. *If* $|\cos(\phi)| = |\cos(\delta)|$, $\tau_1 = \pi/\mu$ *and* $u(0)u(\tau_1) < 0$ *then*

$$S = P|\cos(\phi)|\coth(\gamma\pi/4m\mu).$$

Further Explorations

One can simulate these results by creating various examples in an appropriate software. Pleym [3] illustrated animations of harmonic oscillations by the Maple V software and the programs written for Maple V can be adapted to current Maple versions. Our work was done with Maple 17. A proof of **Theorem 1** can be found in Singh [5].

References

[1] Basavaraju, G. and D. Ghosh 1984. *Mechanics and Thermodynamics*. India: Tata McGraw-Hill.

[2] Boyce, W. E. and R. C. DiPrima 2009. *Elementary Differential Equations and Boundary Value Problems, Ninth Edition*. New York: Wiley.

[3] Pleym, Herald, 1999. Visualizing Free and Forced Harmonic Oscillations using Maple V R5*. *Int. J. Engng. Ed.* 15(6):437–455.

[4] Farlow, J. Stanley, 1994. *An Introduction to Differential Equations and their Applications.* Dover Publications Inc.

[5] Singh, S . 2019. A Sinusoidal Twist with Exponential Influences. *Primus.* 29(6): 541–552