Birational products

Hans Schoutens
Mathematics

Abstract

Two algebro-geometric objects (aka varieties) are said to be in the same birational class if "for the most part" they are the same. Although the class of varieties is closed under Cartesian products, the product of two varieties in the same birational class will therefore be "twice" as big. Is there a way to refine the product so that the result is again in the same birational class? That is the problem of birational products, and I will explain in this poster how it can be solved.

Varieties

- a *variety* is the solution set of a (prime) system of polynomials (over a given field, say, the reals $\mathbb{R}$);
- the *dimension* of a variety, is the number of freedoms on the variety;
- the *function field* of a variety is the set of all partial functions that are undefined on a smaller dimensional subvariety;
- two varieties belong to the same *birational class*, if after taking away a smaller dimensional subvariety in both, they become the same (isomorphic).

Birationality

Corollary

Two varieties belong to the same birational class if and only if they have the same function field.

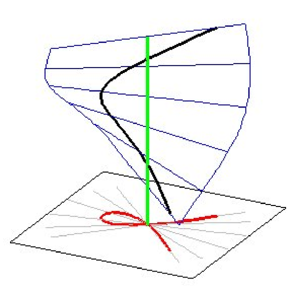
Corollary

There is a one-one correspondence between fields and birational classes.

Corollary

All varieties in a birational class have the same dimension.

Example (The birational class of a parabola)



Except for the points on the green line, the black parabola and the red node with equation $y^2 = x^3 + x^2$ are isomorphic, whence in the same birational class. In spite of the similarity of its equation, the elliptic curve with equation $y^2 = x^3 + x$ is not in that class.

Cartesian Products

Definition (Cartesian Product)

The *Cartesian* (aka, *fiber*) product $V \times W$ of two varieties V and W consists of all pairs of points (P, Q) with $P \in V$ and $Q \in W$.

Corollary

The dimension of $V \times W$ is the sum of the dimensions of V and W.

Therefore, if V and W are in the same birational class, $V \times W$ is **not**.

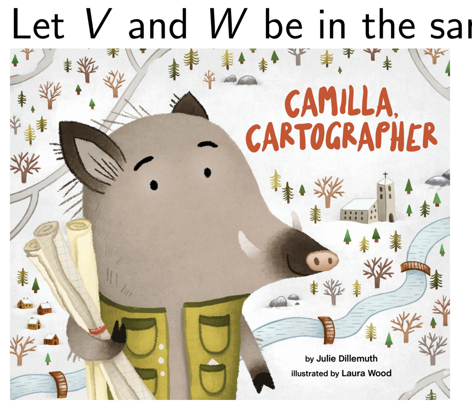
Towards a birational product

Question

How to define the product on a birational class?

Idea

Let V and W be in the same class and let U be their common part.



- Decompose V as $U \cup V'$, with V' the extra part of lesser dimension;
- Decompose W as $U \cup W'$, with W' the extra part of lesser dimension;
- Put together U with V' and W'.
- But V and W may live in different 'worlds', so how to do this last step?
- We need a cartographer!

Grounding varieties

- First, we have to find a common ground, and by the above corollary, that is given by the function field K.
- We say that a variety V is *grounded* if we fix how the elements of K act as (partial) functions on V.
- But functions have denominators and what if they become zero, for instance y/x at the point $(0, 0)$.
- We need a strategy to calculate limits and this can be done using *valuations*.

Valuations

Definition

A *valuation* on a field K is a (quantitative) way of deciding the following two questions:

1. Given a variety V with function field K, does it have a center $P \in V$?
2. If yes, given a function $f \in K$, does $\lim_P f$ converge in K?

Example

Consider the Cartesian plane: what is $\lim_{(0,0)} \frac{y}{x}$? It depends on the valuation! Consider the linear valuation along the line $y = x$: we must set $x = t$ and $y = t$ and take the limit for t to 0, and so the limit is $\lim_{t \to 0} \frac{t}{t} = 1$.
In contrast, the cuspidal valuation which approaches $(0, 0)$ along the cusp $x^2 - y^3 = 0$ amounts to setting $x := t^3$ and $y := t^2$, and now the limit is $\lim_{t \to 0} t^2/t^3 = \lim_{t \to 0} 1/t = \infty$.

Definition (Complete varieties)

A variety V is *complete* if (1) always holds.
The projective plane (see below) is an example of a complete variety.

Atlases and the Zariski-Riemann space

Definition

- A *chart* is a collection $\mathfrak{U}(f_1, \ldots, f_s)$ of valuations that are convergent on some functions $f_1, \ldots, f_s \in K$.
- The *Zariski-Riemann space* $\mathbf{ZR}(K)$ of a function field K is the set of all valuations on K with the *chart topology*.
- An *atlas* is a collection of compatible charts that cover the entire space $\mathbf{ZR}(K)$.
- Two atlases are called *similar* if they have the same 'stalks' on $\mathbf{ZR}(K)$.
- One atlas *refines* another, if any chart in the first is contained in a chart of the second.

Construction (Affine varieties)

Let $\mathfrak{U}$ be a chart.

- Consider the set A of all $f \in K$ such that each valuation in $\mathfrak{U}$ converges at f.
- The quantitative part of the notion of valuation yields that A is closed under addition and multiplication (aka, a *ring*).
- The collection of ring homomorphisms $A \to \mathbb{C}$ can be given the structure of a variety, and as such, its function field is K.
- We call it the *affine* variety determined by the chart $\mathfrak{U}$.

We can now generalize this construction to arbitrary complete varieties.

Theorem (Pignatti, S.)

Given a birational class with corresponding function field K; there is a one-one correspondence between atlases (up to similarity) and (normal) complete varieties in this class. Moreover, maps between varieties correspond to refinements of atlases.

Projective plane

Example (The birational class of the Cartesian plane)

Let $K := \mathbb{R}(x, y)$; it is the function field of the Cartesian plane. What else is in this birational class?

- Consider the three charts $\mathfrak{U}(x, y)$, $\mathfrak{U}(\frac{1}{x}, \frac{y}{x})$ and $\mathfrak{U}(\frac{x}{y}, \frac{1}{y})$. The first of these corresponds to the Cartesian plane, and together they form an atlas $\mathcal{A}$ with corresponding variety the *projective plane* $\mathbb{P}^2$.
- Now consider the four charts $\mathfrak{U}(x, \frac{y}{x})$, $\mathfrak{U}(\frac{x}{y}, y)$, $\mathfrak{U}(\frac{1}{x}, \frac{y}{x})$ and $\mathfrak{U}(\frac{x}{y}, \frac{1}{y})$, forming the atlas $\bar{\mathcal{A}}$. The corresponding variety is called the *blow-up* $\bar{X}$ of $\mathbb{P}^2$ at the origin.
- Since $\bar{\mathcal{A}}$ is a refinement of $\mathcal{A}$, we get the *blow-up* map $\bar{X} \to \mathbb{P}^2$; it is the map depicted in the picture on the left, and, for instance, it maps the black curve onto the red curve.

Birational products

Main Theorem

Let V and W be complete varieties in the same birational class, and let $\mathcal{A}$ and $\mathcal{B}$ be their respective atlases. Consider the collection $\mathcal{A} \wedge \mathcal{B}$ consisting of all intersections $\mathfrak{V} \cap \mathfrak{W}$, where $\mathfrak{V} \in \mathcal{A}$ and $\mathfrak{W} \in \mathcal{B}$. It is again an atlas, and the complete variety it determines is the birational product of V and W.