The multi-dimensional realm of Feynman Integrals

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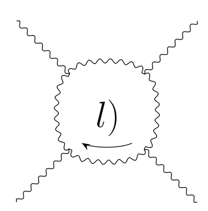
Abstract

The understanding of our Universe at the smallest scales, and the interaction among fundamental particles that populate them, relies on the understanding of their symmetries. The mathematical language that we use to describe such interactions should indeed reflect the same structures.
Within this framework, the study of Feynman integrals allows us to go beyond its traditional role of the standard computational technique in perturbation theory and becomes an extraordinary tool to explore the symmetries of our particle physics models.
In our poster, we discuss various parametrizations of Feynman Integrals and their specific features. By choosing different parametrizations, we do not simply choose different variables to represent the same multi-dimensional integrals, but we explore different facets of the underlying mathematical structures of scattering amplitudes.

Feynman Integrals and Momentum Parametrization

A scattering amplitude is given in a diagram form → Feynman diagrams:



$$\mapsto \int d^d l \frac{N(l)}{D_1 D_2 D_3 D_4}$$

This translation can be achieved by means of Feynman rules associated with the theoretical model. It is well-known that we can decompose a Feynman Integral into a linear combination of scalar integrals. [8, 9, 10, 6, 4] We, therefore, consider a family of L-loops scalar integrals:

$$I(\vec{\nu}) := \prod_{a=1}^{L} \int \frac{d^d l_a}{(2\pi)^d} \frac{1}{D_1^{\nu_1} \cdots D_N^{\nu_N}}.$$

Baikov Parametrization

Under the integration variable change

$$(l_1, \cdots, l_L) \mapsto (D_1, \cdots, D_N),$$

we obtain [2, 4] the following parametrization:

$$I(\vec{\nu}) \sim \int \frac{dD_1 \cdots dD_N}{D_1^{\nu_1} \cdots D_N^{\nu_N}} P^{\frac{d-L-E-1}{2}}$$

where P is the Jacobi determinant of this variable change

$$P = \det [q_i \cdot q_j] (D_1, \cdots, D_N)$$

that is, the determinant of scalar products expressed as a polynomial of denominators, and this P is called the Baikov polynomial. The integration domain is determined by the zeros of P.
P and the integration domain do not depend on the indices $\nu_1, \cdots, \nu_N$, so the family of integrals are characterized by a polynomial P.

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Lee-Pomeransky Parametrization

We can show the scalar integrals becomes

$$I(\vec{\nu}) \sim \prod_{a=1}^{N} \int_0^\infty \frac{dx_a x^{\nu_a - 1}}{\Gamma(\nu_a)} \mathcal{G}^{-d/2}$$

of Lee-Pomeransky parametrization [5], where Lee-Pomeransky polynomial

$$\mathcal{G} := \mathcal{U} + \mathcal{F}$$

and two graph polynomials are defined graph-theoretically; $\mathcal{U}$ is the chord-set product of the underlying graph and $\mathcal{F}$ can be build with 2-tree product and the corresponding inner product of momenta. $\mathcal{G}$ does not depend on indices or momenta labeling, and characterizes the family of integrals.
We can also show that these polynomials can be derived through graph-theoretically, [7] i.e., they depend only on the Feynman graph. These polynomials, indeed, reflect the graph-theoretical symmetry of Feynman graph; for example, if the graph has swap symmetry of internal lines, the polynomials are symmetric under the corresponding parameters.

IBP-like identities over LP

Through the following functional:

$$G_i(f) = \int dx_1 \ldots \int dx_N \, \partial_i \left[f \mathcal{G}^{-\frac{d}{2}+1} \right]$$

we can build a set of linear equations: on the one hand, we apply the partial derivative to obtain a linear combination of integrals with full N denominators:

$$G_i(f) = \int dx_1 \ldots \int dx_N \, \underbrace{\partial_i \left[f \mathcal{G}^{-\frac{d}{2}+1} \right]}_{\text{partial derivative}},$$

On the other hand, we use the fundamental theorem of calculus to associate this functional with the surface term; under dimensional regularization scheme, we can identify this surface term as another linear combination of integrals that has no D_i:

$$G_i(f) = \left(\prod_{a \neq i} \int dx_a \right) \underbrace{\int dx_i \partial_i}_{\text{surface term}} \left[f \mathcal{G}^{-\frac{d}{2}+1} \right].$$

Other parametrizations, e.g., the defining momentum parametrization and Baikov parametrization, have zero surface-terms. Lee-Pomeransky parametrization (in general Schwinger-Feynman-Lee-Pomeransky parametrizations) have, in general, non-zero surface term [1, 3]

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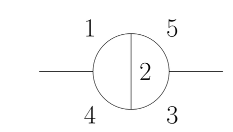
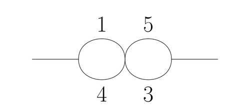
Non-positive Indices with CIF and Dimension Shift

To better formulate the problem, let us begin by comparing the "kite" diagram:



$$\sim \int \frac{1}{D_1 D_2 D_3 D_4 D_5}.$$

with the "glasses" diagram:



$$\sim \int \frac{1}{D_1 D_3 D_4 D_5}.$$

If we are to write this glasses integral in LP parametrization, we obtain

$$I_{\text{glasses}}(1,1,1,1) \sim \left(\prod_{a \in \{1,3,4,5\}} \int_{x_a=0}^{\infty} dx_a \right) \mathcal{G}_{\text{glasses}}^{-\frac{d}{2}},$$

where $\mathcal{G}_{\text{glasses}}$ is defined through the four denominator configuration. The same integral can be obtained by deleting D_2 in the integrand of the kite diagram, i.e., by setting the index $\nu_2 = 0$, and, therefore, we can claim that the diagram belongs to the kite family:

$$I(\nu_1, \nu_2, \nu_3, \nu_4, \nu_5) \sim \prod_{a=1}^{5} \int_{x_a=0}^{\infty} \frac{dx_a x_a^{\nu_a - 1}}{\Gamma(\nu_a)} \mathcal{G}_{\text{kite}}^{-\frac{d}{2}}.$$

However, if we were to set $\nu_2 = 0$ in the expression above, we would immediately face some ill-behaved factors, namely the appearance of a factor $1/\Gamma(0)$ in front of our integral and the presence of ${x_2}^{-1}$ in the integrand, which would lead to a divergent integral in x_2:

$$I(1,0,1,1,1) \overset{?}{\sim} \left(\prod_{a \in \{1,3,4,5\}} \int_{x_a=0}^{\infty} dx_a \right) \int_{x_2=0}^{\infty} \frac{dx_2 {x_2}^{-1}}{\Gamma(0)} \mathcal{G}_{\text{kite}}^{-\frac{d}{2}}.$$

Observe the fact $\mathcal{G}_{\text{kite}}|_{x_2=0} = \mathcal{G}_{\text{glasses}}$ and it leads the followings; if we change he integration domain from the positive real axis to an infinitesimal small anti-clockwise circle around the origin in the complex plane:

$$\int \frac{dx_2 x_2^{-1}}{\Gamma(0)} \mathcal{G}_{\text{kite}} \mapsto \frac{1}{2\pi i} \oint \frac{dx_2}{x_2} \mathcal{G}_{\text{kite}} = \mathcal{G}_{\text{kite}}|_{x_2=0} = \mathcal{G}_{\text{glasses}}.$$

This is nothing but Cauchy's integral formula.

We can further generalize this approach to represent diagrams in which a denominator D appears with negative indices ν, i.e., sits in the numerator of the Feynman Integral with power $n = -\nu$. Relying again on Cauchy's integral formula,

$$D^n = (-)^n \frac{n!}{2\pi i} \oint \frac{dx}{x^{n+1}} \exp(-xD) = \left(\frac{\partial}{\partial x} \right)^n \exp(-xD) \Big|_{x=0},$$

which allows us to write the following replacement rule:

$$\int \frac{dx_a}{\Gamma(-n) x_a^{n+1}} \mathcal{I} \mapsto (-)^n \frac{n!}{2\pi i} \oint \frac{dx_a}{x_a^{n+1}} \mathcal{I} = (-)^n \frac{\partial^n}{\partial x_a^n} \mathcal{I} \Big|_{x_a=0},$$

where $\mathcal{I}$ stands for an arbitrary integrand.

As an example of application, let us consider:

$$I_{\text{kite}}(1,-1,1,1,1) \overset{?}{\sim} \left(\prod_{a \in \{1,3,4,5\}} \int_{x_a=0}^{\infty} dx_a \right) \int_{x_2=0}^{\infty} \frac{dx_2 \, {x_2}^{-2}}{\Gamma(-1)} \mathcal{G}_{\text{kite}}^{-\frac{d}{2}},$$

which again contains worrisome features, such as the appearance of the factor $1/\Gamma(-1)$ and the presence of the diverging ${x_2}^{-2}$ in the integrand. However, employing the replacement rule, we can set

$$\int_{x_2=0}^{\infty} \frac{dx_2 \, {x_2}^{-2}}{\Gamma(-1)} \mathcal{G}_{\text{kite}}^{-\frac{d}{2}} \mapsto (-1) \frac{\partial}{\partial x_2} \mathcal{G}_{\text{kite}}^{-\frac{d}{2}} \Big|_{x_2=0},$$

which, thanks to the fact that $\mathcal{G}_{\text{kite}}|_{x_2=0} = \mathcal{G}_{\text{glasses}}$, leads to a well-defined linear combination of integrals in the family of $\mathcal{G}_{\text{glasses}}$.

After applying ∂_2 on $\mathcal{G}_{\text{kite}}^{-\frac{d}{2}}$, it reduces the power $-\frac{d}{2}$ by one and the outcome becomes as a polynomial of (x_1, x_3, x_4, x_5) times $\mathcal{G}_{\text{glasses}}^{-\frac{d}{2}-1} = \mathcal{G}_{\text{glasses}}^{-\frac{d+2}{2}}$. Thus, we can identify them as a linear combination of integrals in a shifted dimension $d \mapsto d+2$.