Rédei Permutations with Cycles of the Same Length

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Abstract

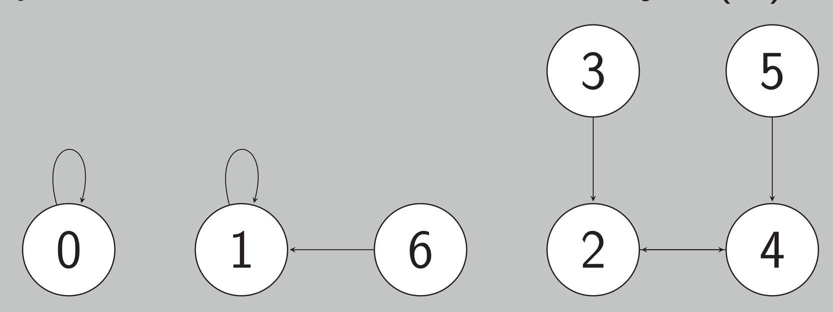
Let $\mathbb{F}_q$ be a finite field of odd characteristic. We study Rédei functions that induce permutations over $\mathbb{P}^1(\mathbb{F}_q)$ whose cycle decomposition contains only cycles of length 1 and j, for an integer $j \geq 2$. When j is a prime number, we give necessary and sufficient conditions for a Rédei permutation of this type to exist over $\mathbb{P}^1(\mathbb{F}_q)$, characterize Rédei permutations consisting of 1- and j-cycles, and determine their total number. We also present explicit formulas for Rédei involutions based on the number of fixed points.

Functional Graphs

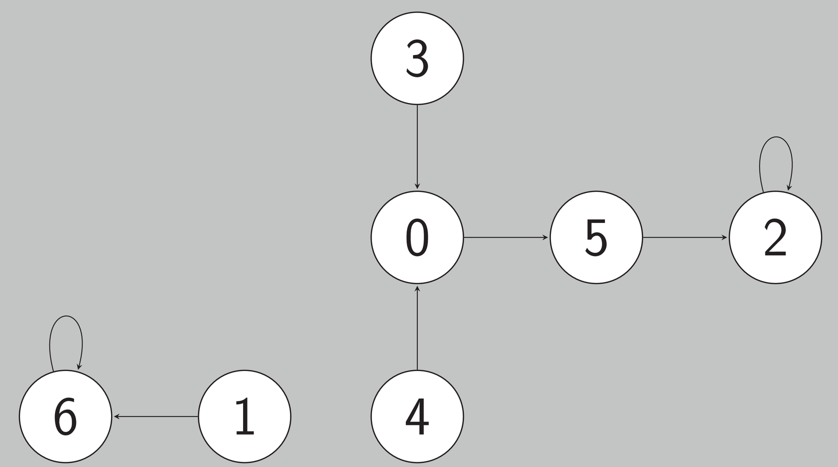
- Let S be a finite set and $f : S \to S$ be a mapping.
- The **functional graph associated to f** is a directed graph where the vertices are labelled by the elements of S, and a directed edge connects a vertex a with a vertex b if and only if $b = f(a)$.
- Let $\mathbb{F}_q$ be the finite field of order q.

Example

▷ The functional graph of $f : \mathbb{F}_7 \to \mathbb{F}_7$ defined by $f(x) = x^2$ is



▷ The functional graph of $g : \mathbb{F}_7 \to \mathbb{F}_7$ defined by $g(x) = x^2 + 5$ is



The functional graphs of x^2 and $x^2 + 5$ are not isomorphic.

Rédei Function

- Let $\mathbb{P}^1(\mathbb{F}_q) := \mathbb{F}_q \cup \{\infty\}$.
- Write $(x + \sqrt{y})^m$ as $N(x, y) + D(x, y)\sqrt{y}$.
- For a positive integer m and $a \in \mathbb{F}_q$, the **Rédei function** is $R_{m,a} : P^1(\mathbb{F}_q) \to P^1(\mathbb{F}_q)$ where

$$R_{m,a}(x) = \begin{cases} \dfrac{N(x,a)}{D(x,a)} & \text{if } D(x,a) \neq 0, x \neq \infty \\ \infty & \text{otherwise.} \end{cases}$$

The Isomorphism Problem

- We denote the functional graph of $R_{m,a}$ over $\mathbb{P}^1(\mathbb{F}_q)$ by $\mathcal{G}(m, a, q)$.
- **Problem**: Find conditions on m, n, a, b, q such that $\mathcal{G}(m, a, q)$ is isomorphic to $\mathcal{G}(n, b, q)$.
- Our goal is to characterize permutation $R_{m,a}$ with 1- and j-cycles for a prime j.

The Graph Structure

- $\chi(a) = 1$ if a is a square in $\mathbb{F}_q^*$, and -1 otherwise.

Theorem (Qureshi and Panario [2])

The Rédei function $R_{m,a}$ induces a permutation of $\mathbb{P}^1(\mathbb{F}_q)$ if and only if $\gcd(m, q - \chi(a)) = 1$. In this case, we have the following decomposition in disjoint cycles:

$$\mathcal{G}(m, a, q) \cong \bigoplus_{d \mid q-\chi(a)} \left\{ \frac{\phi(d)}{o_d(m)} \times Cyc(o_d(m)) \right\} \oplus (1 + \chi(a)) \times \{\bullet\},$$

where ϕ is the Euler's totient function, $o_d(m)$ is the order of m modulo d, and $Cyc(c)$ denotes a c-cycle.

Main Results

Theorem (Capaverde, M., and Rodrigues [1], 2020)

Let p be an odd prime. There exists a Rédei permutation over $\mathbb{P}^1(\mathbb{F}_q)$ with cycles of length 1 and p if and only if $q - 1$ or $q + 1$ has a prime factor of the form $pk + 1$ or is divisible by p^2.

Theorem (Capaverde, M., and Rodrigues [1], 2020)

Let p be an odd prime and M be the number of Rédei permutations over $\mathbb{P}^1(\mathbb{F}_q)$ with cycles of length 1 and p with fixed parameter a. Then

$$M = \begin{cases} p^r - 1 & \text{if } p^2 \nmid q - \chi(a) \\ p^{r+1} - 1 & \text{if } p^2 \mid q - \chi(a), \end{cases}$$

where $r = |\{p' \text{ prime} \colon p' \equiv 1 \pmod{p}, p' \mid q - \chi(a)\}|$.

Theorem (Capaverde, M., and Rodrigues [1], 2020)

Let $\nu_p(z)$ be the p-adic valuation of z. A Rédei permutation $R_{m,a}$ over $\mathbb{P}^1(\mathbb{F}_q)$ is an involution with $d + \chi(a) + 1$ fixed points if and only if d is even, $\nu_2(d) \in \{1, \nu_2(q - \chi(a)) - 1, \nu_2(q - \chi(a))\}$, and $\gcd(d, (q - \chi(a))/d) \mid 2$. In this case, $m \equiv k(q - \chi(a))/d - 1 \pmod{q - \chi(a)}$, where k reduced modulo d equals

$$\begin{cases} 2\left(\frac{q-\chi(a)}{d}\right)^{\varphi(d)-1} & \text{if } \nu_2(d) = \nu_2(q - \chi(a)) \\ \left(\frac{q-\chi}{2d}\right)^{\varphi(d)-1} + \frac{d}{2} & \text{if } \nu_2(d) = \nu_2(q - \chi(a)) - 1 \geq 1 \\ \left(\frac{q-\chi(a)}{2d}\right)^{\varphi(d)-1}, \left(\frac{q-\chi(a)}{2d}\right)^{\varphi(d)-1} + \frac{d}{2} & \text{if } \nu_2(d) = 1, \nu_2(q - \chi(a)) \geq 3 \end{cases}$$

Remarks

- The type of function under investigation is of interest in the construction of interleavers for turbo codes. In addition, involutions have cryptographic applications such as the design of S-boxes.
- In our paper, we also give procedures to construct Rédei permutations with a prescribed number of fixed points and j-cycles for $j \in \{3, 4, 5\}$.
- Our results allow us to find all Rédei functions whose functional graphs consist of fixed points and j-cycles where j is any prime number, without the aid of a computer, depending on the factorization of $q \pm 1$.

Example

Let $R_{m,a}$ be a Rédei permutation over $\mathbb{F}_{125}$. The following are **all Rédei permutations with 1- and j-cycles over $\mathbb{P}^1(\mathbb{F}_{125})$, where j is prime**.

▷ when $\chi(a) = 1$: $R_{123,a}$ has 4 fixed points and 61 2-cycles; $R_{61,a}$ has 6 fixed points and 60 2-cycles; $R_{63,a}$ has 64 fixed points and 31 2-cycles; $R_{m,a}$ has 6 fixed points and 40 3-cycles when $m \in \{5, 25\}$; $R_{m,a}$ has 6 fixed points and 24 5-cycles when $m \in \{33, 97, 101, 109\}$.

▷ when $\chi(a) = -1$: $R_{125,a}$ has 2 fixed points and 62 2-cycles; $R_{71,a}$ has 14 fixed points and 56 2-cycles; $R_{55,a}$ has 18 fixed points and 54 2-cycles; $R_{m,a}$ has 6 fixed points and 40 3-cycles when $m \in \{25, 67, 79, 121\}$; $R_{m,a}$ has 18 fixed points and 36 3-cycles when $m \in \{37, 109\}$; $R_{m,a}$ has 42 fixed points and 28 3-cycles when $m \in \{43, 85\}$.

Open Problems

- Find all Rédei permutations with 1- and j-cycles when j is not prime
- Obtain closed formulas for m such that $R_{m,a}$ is a Rédei permutation with 1- and j-cycles, $j \neq 2$.

References

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Acknowledgement. The second author received support for this project provided by a PSC-CUNY Grant.