Tidal locking and gravitational fold catastrophe

Andrea Ferroglia and Miguel Fiolhais

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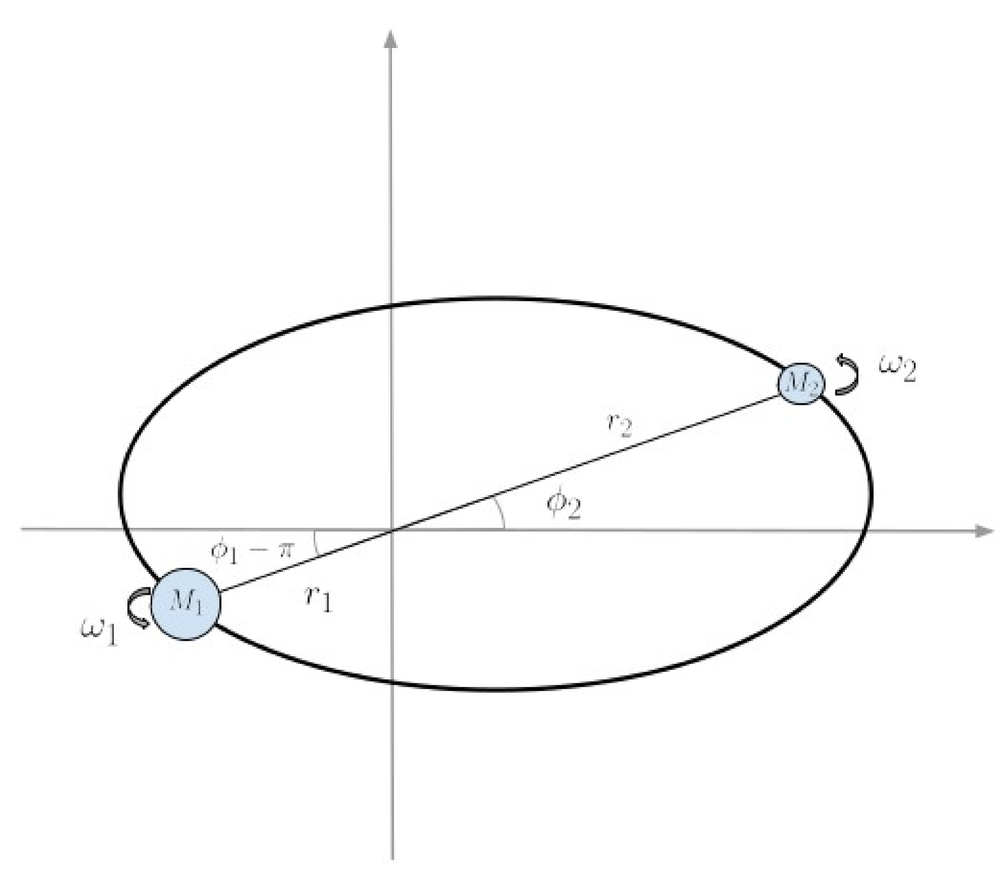
ABSTRACT

The purpose of this work is to study the phenomenon of tidal locking in a pedagogical framework by analyzing the effective gravitational potential of a two-body system with two spinning objects. It is shown that the effective potential of such a system is an example of a fold catastrophe. In fact, the existence of a local minimum and saddle point, corresponding to tidally-locked circular orbits, is regulated by a single dimensionless control parameter which depends on the properties of the two bodies and on the total angular momentum of the system. The method described in this work results in compact expressions for the radius of the circular orbit and the tidally-locked spin/orbital frequency. The limiting case in which one of the two orbiting objects is point-like is studied in detail. An analysis of the effective potential, which in this limit depends on only two parameters, allows one to clearly visualize the properties of the system. The notorious case of the Mars' moon Phobos is presented as an example of a satellite that is past the no return point and, therefore, will not reach a stable or unstable tidally-locked orbit.



The two-body system can be reach a circular stable tidally -locked orbit if the effective potential below has a local minimum.

$$U_{\rm eff} = \frac{G^2 M_1^3 M_2^3}{L^2 (M_1+M_2)} \left[\frac{1}{2\tilde{r}_2^2} - \frac{1}{\tilde{r}_2} - \frac{k_1 \tilde{\omega}_1}{\tilde{r}_2^2} - \frac{k_2 \tilde{\omega}_2}{\tilde{r}_2^2} + \frac{k_1 \tilde{\omega}_1^2}{2} + \frac{k_2 \tilde{\omega}_2^2}{2} + \frac{1}{2}\left(\frac{k_1 \tilde{\omega}_1 + k_2 \tilde{\omega}_2}{\tilde{r}_2} \right)^2 \right] .$$

where

$$\tilde{r}_2 \equiv \frac{G M_1 M_2^2 r_2}{L^2}, \quad \tilde{\omega}_1 \equiv \frac{L^3 (M_1+M_2)\omega_1}{G^2 M_1^3 M_2^3}, \quad \tilde{\omega}_2 \equiv \frac{L^3 (M_1+M_2)\omega_2}{G^2 M_1^3 M_2^3}$$

$$k_1 \equiv \frac{G^2 M_1^3 M_2^3 I_1}{L^4 (M_1+M_2)}, \quad k_2 \equiv \frac{G^2 M_1^3 M_2^3 I_2}{L^4 (M_1+M_2)}$$

The potential depends on the two parameters k, which in turn depend on the total angular momentum L, on the masses of the two objects, M, and on their moments of inertia I. The potential has a local minimum if and only if

$$k_1 + k_2 \leq \frac{27}{256}$$

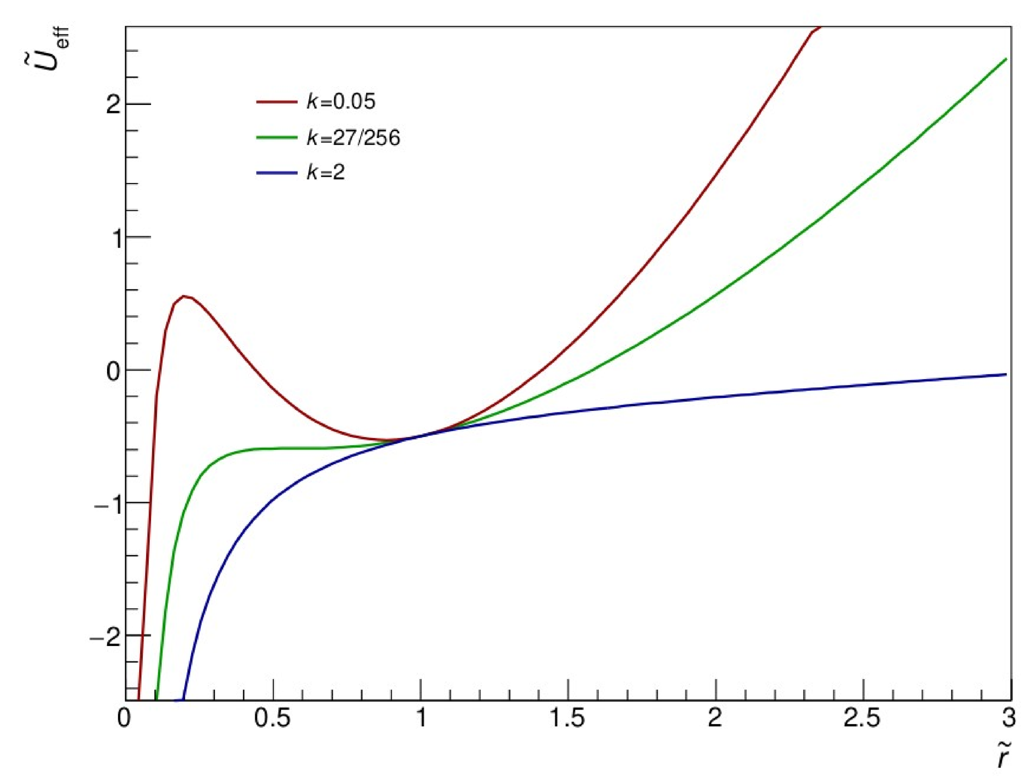
Condition for the presence of a stable, circular tidally locked orbit

CIRCULAR ORBIT

The analysis becomes simpler and easier to visualize when one considers a point like satellite. If in addition one imposes to be in a circular orbit, the effective potential that needs to be minimized in order to a have a stable tidally locked configuration becomes

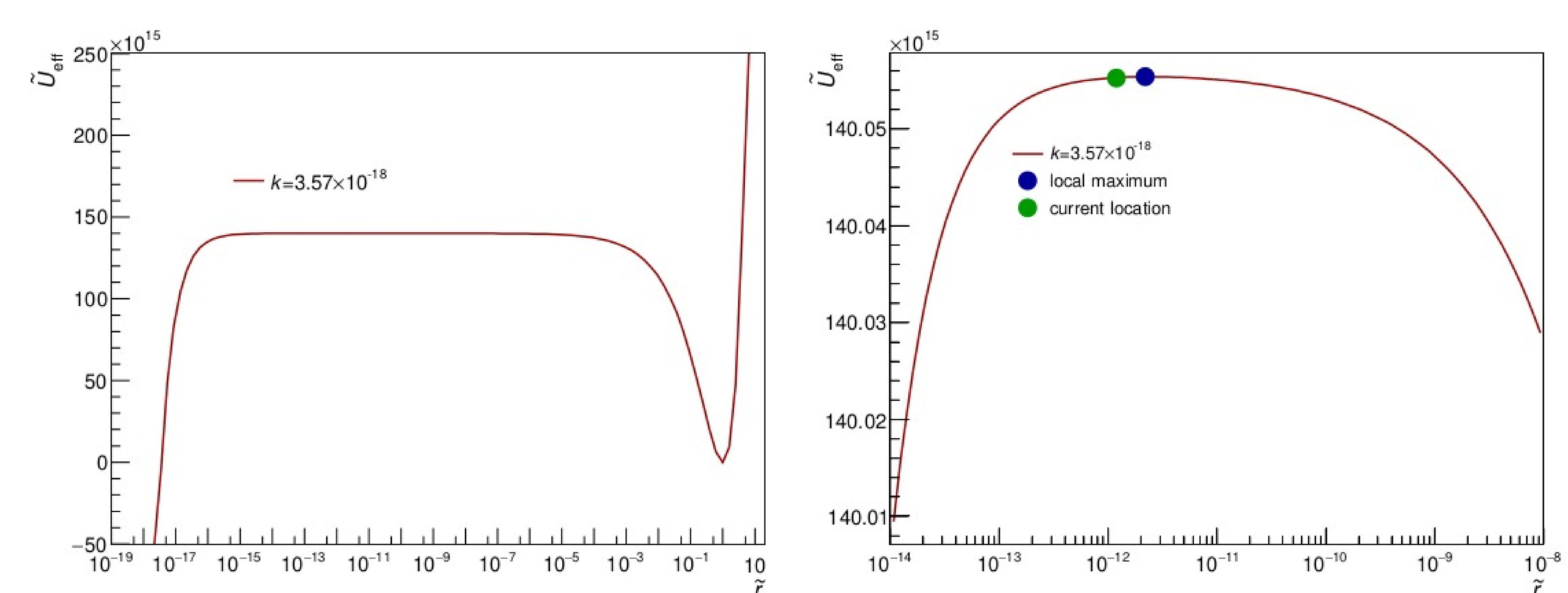
$$U_{\rm eff}(\tilde{r}) \equiv \frac{G^2 M^2 m^3}{L^2} \tilde{U}_{\rm eff}(\tilde{r}) = \frac{G^2 M^2 m^3}{L^2} \left[\frac{1}{2k} - \frac{1}{2\tilde{r}} - \frac{\sqrt{\tilde{r}}}{k} + \frac{\tilde{r}}{2k} \right]$$

Again, this potential has a minimum only for k < 27/256. The shape of the effective potential above and below the critical value of k is shown below

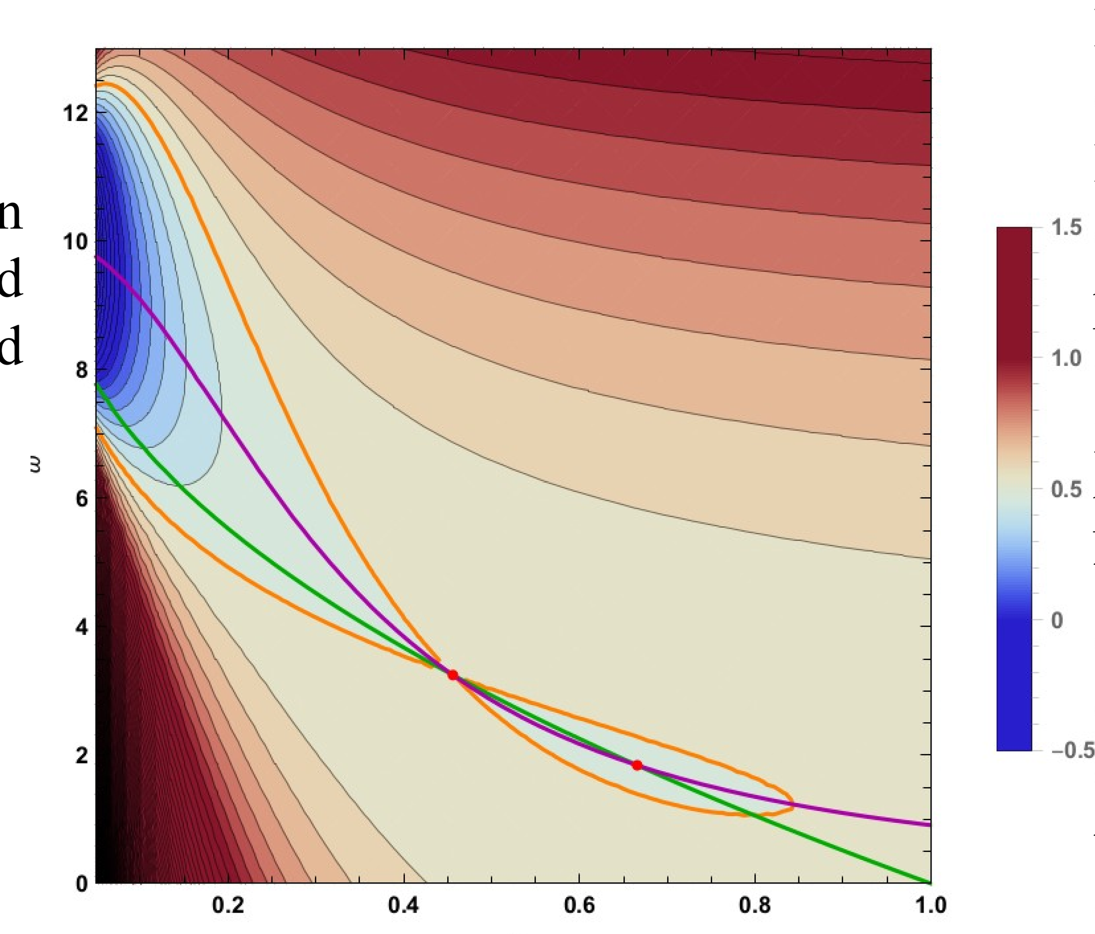


This phenomenon can be interpreted as a fold catastrophe. In catastrophe theory, a fold catastrophe represents a system where a single control parameter can cause equilibrium points to appear or disappear.

With this analysis one can show that the the Mars satellite Phobos will eventually fall on Mars. Indeed the satellite has a value of k well below the critical value of 27/256, so a stable tidally locked orbit does exist. However, while the system Phobos Mars loses mechanical energy due to tidal friction, the current radial distance of Phobos from Mars is to the left of the local maximum of the effective potential. Therefore Phobos cannot reach the stable circular tidally locked orbit, and it will eventually fall on Mars.



ELLIPTIC ORBIT



It is possible to analyze the potential in the case in which one allows for elliptical orbits. In that case one needs to study a two dimensional effective potential. This potential will have a local minimum and a saddle point below the critical value k < 27/256. The green curve corresponds to points in the $\{\tilde{r}, \tilde{\omega}\}$ plane where the orbit is tidally locked. The magenta curve corresponds to points where the satellite's orbit is circular. The local minimum (rightmost red dot) corresponds to a circular, tidally locked orbit. The same is true for the saddle point (leftmost red dot).