

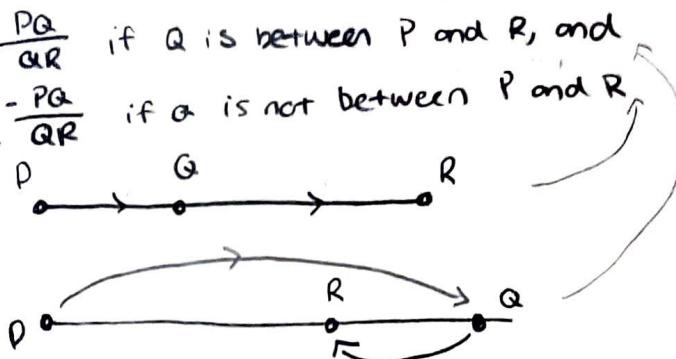
=> BACK TO CHPT 8

(Can someone tell us what this means)

Def Assume P, Q, and R are three distinct collinear points. Define the sensed ratio  $PQ/QR$  by

$$\rightarrow \frac{PQ}{QR} = \begin{cases} \frac{PQ}{QR} & \text{if } Q \text{ is between } P \text{ and } R, \text{ and} \\ -\frac{PQ}{QR} & \text{if } Q \text{ is not between } P \text{ and } R \end{cases}$$

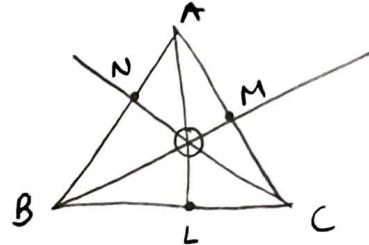
(Notice how this is in bold, it is used to distinguish the sensed ratio from the unsensed ratio.)



### => CEVA'S THEOREM

Let  $\triangle ABC$  be an ordinary triangle. The Cevian lines,  $\overleftrightarrow{AL}$ ,  $\overleftrightarrow{BM}$  and  $\overleftrightarrow{CN}$  are concurrent if and only if

$$\frac{AN}{NB} \cdot \frac{BL}{LC} \cdot \frac{CM}{MA} = +1$$



### => 9.2 DUALITY

In Chpt 8, we studied the problem of determining when three lines through the vertices of a triangle are concurrent.

In Chpt 9, we study the problem of determining when three points on the sidelines of a triangle are collinear.

\*Therefore, the relationship of these two problems is an example of duality.

=> The principle of duality asserts that any true statement in geometry should remain true when the words point and line are interchanged  
\*(point out how point and line are interchanged)\*

Ex. Just as two points lie on exactly one line  $\rightarrow$  two lines intersect in exactly one point.

Just as three points may be collinear  $\rightarrow$  three lines may be concurrent

\*The theorem of Ceva and Menelaus will be our primary example of dual theorems

Ceva rediscovered the theorem of Menelaus and then discovered his own theorem by applying the principle of duality.

## 9.2 The Theorem of Menelaus

\* Before we prove the theorem, let's take a look at some exercises using Geogebra.

9.2.1. Construct a triangle  $\Delta ABC$  and proper Menelaus points  $L, M$  and  $N$  for  $\Delta ABC$ . Calculate the quantity.

$$d = \frac{AN}{NB} \cdot \frac{BL}{LC} \cdot \frac{CM}{MA}$$

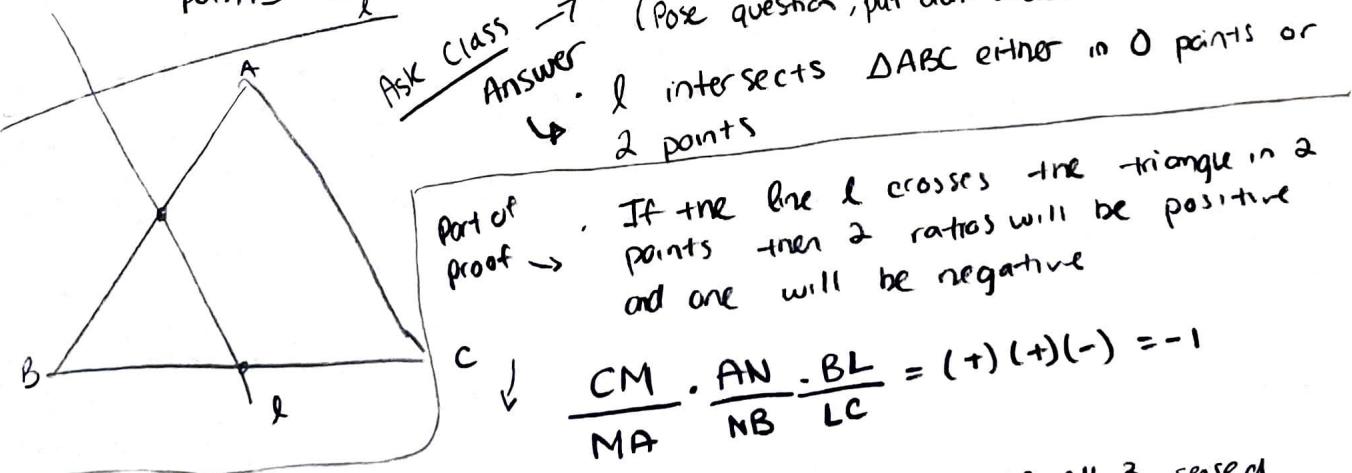
Move the vertices of the triangle and the Menelaus points to verify that if the Menelaus points are collinear, then  $d=1$ .

(Demonstrate Drag Test)

(State that in order to construct this, you have to do sidelines, points on sidelines, then polygon).

9.2.3 Find an example of a triangle and Menelaus points such that  $d=1$  even though the Menelaus points are not collinear.  
maybe for now

9.2.4. Construct a triangle  $\Delta ABC$  and a line  $l$  that does not pass through any of the vertices of the triangle. Determine the answer to the question: [what are the possible number of points of intersection of  $l$  with the triangle]  
Ask class → (Pose question, put don't answer till proof)



$$\frac{CM}{MA} \cdot \frac{AN}{NB} \cdot \frac{BL}{LC} = (+)(+)(-) = -1$$

If the line  $l$  crosses the triangle in 0 points then all 3 sensed ratios will be negative.

$$\frac{CM}{MA} \cdot \frac{AN}{NB} \cdot \frac{BL}{LC} = (-)(-)(-) = -1$$

## STATEMENT OF MENELAUS THM

### ⇒ STATEMENT 1:

Let  $\triangle ABC$  be an ordinary triangle with point  $L$  on  $\overleftrightarrow{BC}$ ,  $M$  on  $\overleftrightarrow{AC}$ ,  $N$  on  $\overleftrightarrow{AB}$ . Assume  $L, M, N$  are collinear.

Then,

$$\frac{AN}{NB} \cdot \frac{BL}{LC} \cdot \frac{CM}{MA} = -1$$

### ⇒ STATEMENT 2:

Let  $\triangle ABC$  be an ordinary triangle with point  $L$  on  $\overleftrightarrow{BC}$ ,  $M$  on  $\overleftrightarrow{AC}$ ,  $N$  on  $\overleftrightarrow{AB}$

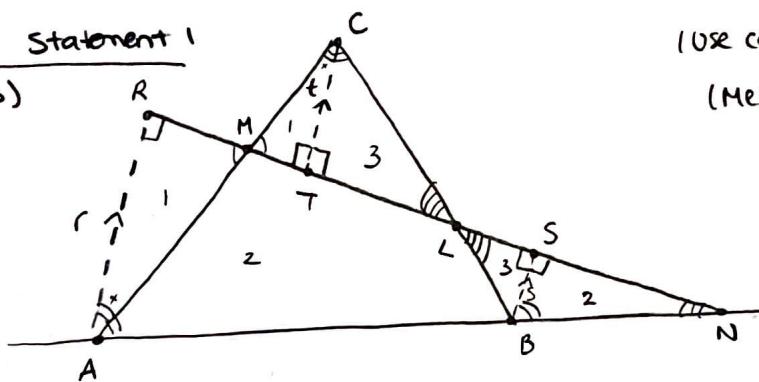
Assume

$$\frac{AN}{NB} \cdot \frac{BL}{LC} \cdot \frac{CM}{MA} = -1$$

Then  $L, M, N$  are collinear

### PF of Statement 1

(no signs)



(use colors to label triangle)

(mention  $\cong$ )

proportionality -

two quantities are related in linear manner

①  $\angle CTM \cong \angle ARM$

$\angle RMA \cong \angle TMC$  by vertical opp  $\angle$ 's  
 $\angle RAM \cong \angle TCM$  since sum of a triangle is  $180^\circ$ , forces  $\angle$  to be  $\cong$  to other

$\triangle CTM \sim \triangle ARM$  by AAA

$$\Rightarrow \text{proportionality} \Rightarrow \frac{CM}{MA} = \frac{CT}{AR} = \frac{t}{r} \leftarrow \text{verema}$$

②  $\angle NRA \cong \angle NSB$

$\angle LN = \angle LN$  by common angle

$\angle NAM \cong \angle NBS$  " "

$\triangle NRA \sim \triangle NSB$  by AAA

$$\Rightarrow \text{proportionality} \Rightarrow \frac{AN}{NB} = \frac{AR}{BS} = \frac{r}{s} \quad (\text{mention sign later})$$

③  $\angle BSL \cong \angle CLT$

$\angle BLN \cong \angle CLT$  by vertical opp  $\angle$ 's

$\angle SBL \cong \angle TCL$  " "

$$\triangle BSL \sim \triangle CTL \Rightarrow \text{proportionality} \Rightarrow \frac{BL}{LC} = \frac{BS}{CT} = \frac{s}{t}$$

\*ignoring signs :  $\frac{CM}{MA} \cdot \frac{AN}{NB} \cdot \frac{BL}{LC} = \frac{CT}{AR} \cdot \frac{AB}{BS} \cdot \frac{BS}{CT} = +1$

\* putting signs back in (9.2.4) explains why signs have to be -1)

- If the line  $l$  crosses the triangle in 2 points then 2 signed ratios will be positive and 1 will be negative.

$$\frac{CM}{MA} \cdot \frac{AN}{NB} \cdot \frac{BL}{LC} = (+)(+)(-) = -1$$

- If the line  $l$  crosses the triangle in 0 points then all 3 signed ratios will be negative

$$\frac{CM}{MA} \cdot \frac{AN}{NB} \cdot \frac{BL}{LC} = (-)(-)(-) = -1$$

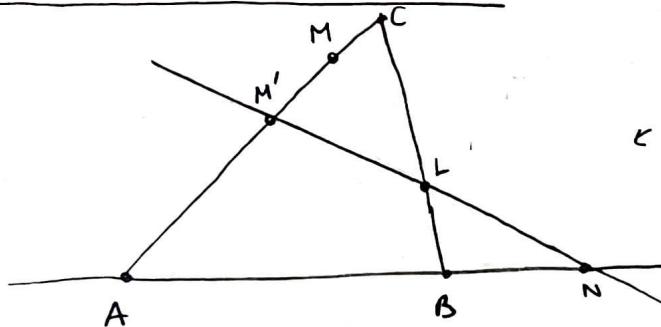
(show on geogebra  
for clarification)

(Mention ②)  $\rightarrow \frac{AN}{NB} = \frac{-AR}{BS} = \frac{-r}{s}$

so,

$$\frac{CM}{MA} \cdot \frac{AN}{NB} \cdot \frac{BL}{LC} = \frac{k}{r} \cdot \frac{-r}{s} \cdot \frac{s}{k} = -1$$

Pf of Statement 2 Uses Statement 1



↖ draw line  $\overleftrightarrow{LN}$ , label intersection of  $\overleftrightarrow{LN}$  and  $\overleftrightarrow{AC}$  by  $M'$

Assume

$$\frac{AN}{NB} \cdot \frac{BL}{LC} \cdot \frac{CM}{MA} = -1$$

We want to show that  $M, L$  and  $N$  are collinear.  
Now suppose that  $M'$  is the intersection of  $\overleftrightarrow{LN}$  and  $\overleftrightarrow{AC}$   
Statement 1 says that

$$\frac{AN}{NB} \cdot \frac{BL}{LC} \cdot \frac{CM'}{MA} = -1$$

(on board write  
on separate  
lines  $\Rightarrow$ )

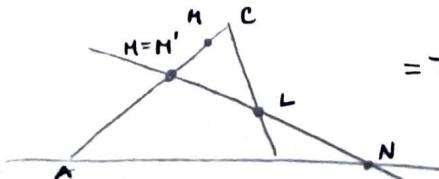
Therefore

$$\frac{AN}{NB} \cdot \frac{BL}{LC} \cdot \frac{CM}{MA} = \frac{AN}{NB} \cdot \frac{BL}{LC} \cdot \frac{CM'}{MA} \Rightarrow \frac{CM}{MA} = \frac{CM'}{MA}$$

$$\Rightarrow \frac{CM}{MA} + 1 = \frac{CM'}{MA} + 1 \Rightarrow \frac{CM+MA}{MA} = \frac{CM'+MA}{MA} \Rightarrow M = M'$$

↖  $CM' + MA = CA = CM + MA$

$M$  and  $M'$  are  
on  $\overleftrightarrow{AC}$  and same  
distance from  
 $A$ .



$\Rightarrow L, N$ , and  $M$  are collinear.