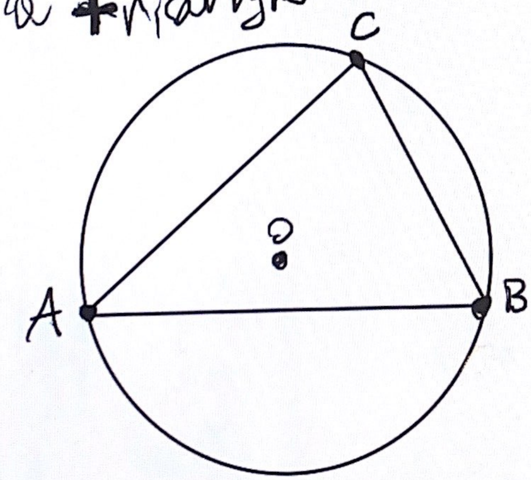
Chapter 4

Circumscribed, Inscribed and Escribed Circles

4.1 circumscribed circle and the circumcenter

Def: A circle that contains all three verticies of a triangle ΔABC is said to circumscribe the triangle. The circle is called the circumscribed circle or simply circumcircle of the triangle. The radius of the circumscribed circle is called the circumradius.

Note: Smallest circle that contains a triangle



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If we look at Ex 2.4.4 it shows that the verticles of a triangle are all equidistant from the circumcenter so they all lie on a cirlce centered on - at the circumcenter. It follows that every triangle has a circumcircle and the circumcenter is its center

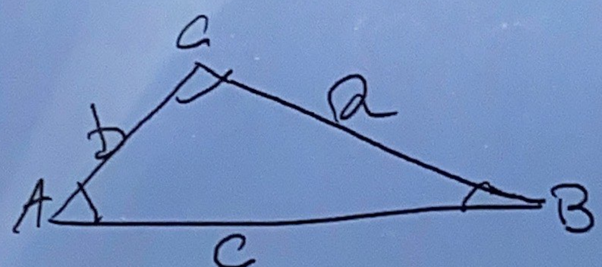
THM: Every triangle has a unique circumscribed circle. The circumcenter is the center of the circumscribed circle

4.1.3 Prove that the circumcenter is unique

Consider three non collinear points A, B, C. (It is possible to draw a circle through the three verticies only if a fourth point exists). The point O is equidistant from A, B, C. Construct a circle surrounding $\triangle ABC$. Any point equidistant from A, B must lie on the perpendicular bisector MN of the side AB. (K1 pg 45 #57). Similarly any point equidistant from B, C must lie on perpendicular bisector PQ at the side BC. Therefore if a point is equidistant from the points A, B and C exists, it must lie on MN and PQ. This is only possible if the point of interfrom the points A, B and C exists, it must lie on MN and PQ. This is only possible if the point of intersection coincides with the intersection point of those lines. The lines MN and PQ do intersect (they are ⊥ to intersecting lines AB and BC (pg 61 tests for non-parallel lines (2)). The intersection point O will be equidistant from A, B and C. Thus, if we take this point for the center, and take the segment OA (or OB, or OC) for radius, then the circle will pass through the points A, B and C. Since the lines MN and PQ can intersect only at one point, the center of such a circle is unique. The length of the radius is also unambiguous, and therefore the circle is unique

Corollary: The point O, being the same distance away from A and C, has to also lie on the perpendicular bisector RS of AC. Thus: three perpendicular bisectors of the sides of a triangle at one point

Law of Sines (high school)



$$\text{Area} = \frac{1}{2}bc\sin A = \frac{1}{2}ba\sin C = \frac{1}{2}ac\sin B$$

$$\frac{b\sin A}{a\sin B} = 1 \qquad \frac{c\sin A}{a\sin C} = 1 \qquad \frac{b\sin C}{c\sin B} = 1$$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Extended Law of Sines

If $\triangle ABC$ is a triangle with circumradius R, then

$$\frac{BC}{\sin(\angle BAC)} = \frac{AC}{\sin(\angle ABC)} = \frac{AB}{\sin(\angle ACB)} = 2R$$

prove this for homework

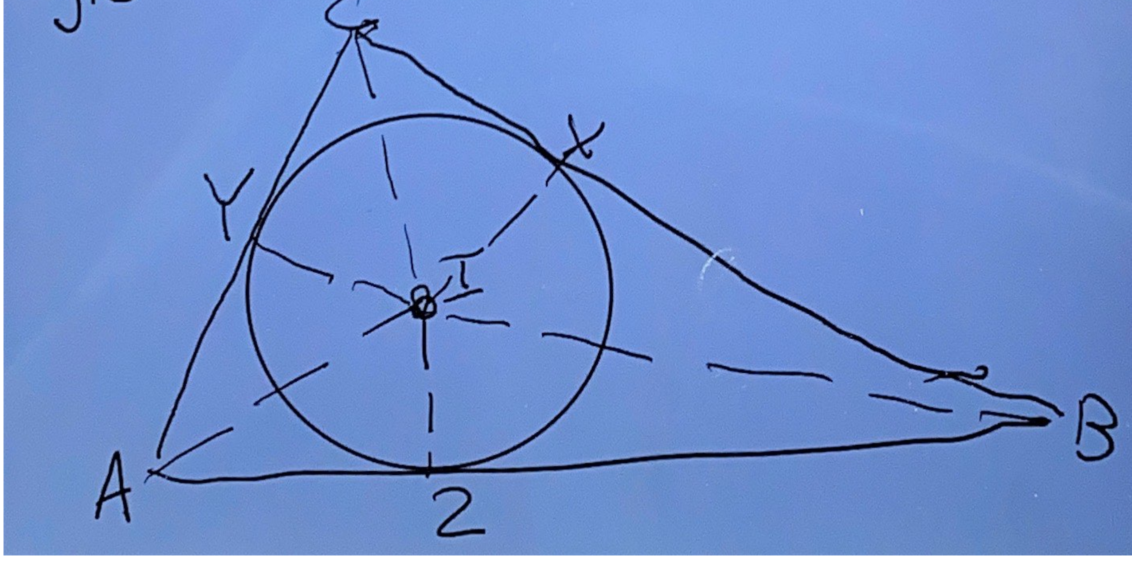
4.2 The Inscribed circle and the incenter

The second circle we we look at is the insc

The second circle we we look at is the inscribed circle or incircle

Note: we want the largest circle we can get to be contained in the triangle

Angle Bisector Concurrence Theorem: If $\triangle ABC$ is any triangle, the bisectors of the interior angles of $\triangle ABC$ are concurrent. The point of concurrency is equidistant from the sides of a triangle



We know that the incenter is always inside the triangle so the $\perp$ bisectors are always inside too.

Th.2.5

Proof: AI and CI are not $\parallel$

$\angle CAI$ and $\angle ACI$ are same side interior angles

For all $\parallel$ lines angles should be supplementary

$\mu(\angle A) + \mu(\angle C) < 180^\circ$

since $\mu(\angle A) + \mu(\angle B) + \mu(\angle C) = 180^\circ$

$\mu(\angle CAI) = \frac{1}{2}\mu(\angle A)$

$\mu(\angle ACI) = \frac{1}{2}\mu(\angle C)$

so the sum is less 180°.

This means that AI is an angle bisector of A ($YI \cong ZI$)

CI is an angle bisector of

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Proof: AI and CI are not $\parallel$ (Th. 2.5)

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For all $\parallel$ lines angles should be supplementary

$\mu(\angle A) + \mu(\angle C) < 180°$

since $\mu(\angle A) + \mu(\angle B) + \mu(\angle C) = 180°$

$\mu(\angle CAI) = \frac{1}{2}\mu(\angle A)$

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so the sum is less $180°$.

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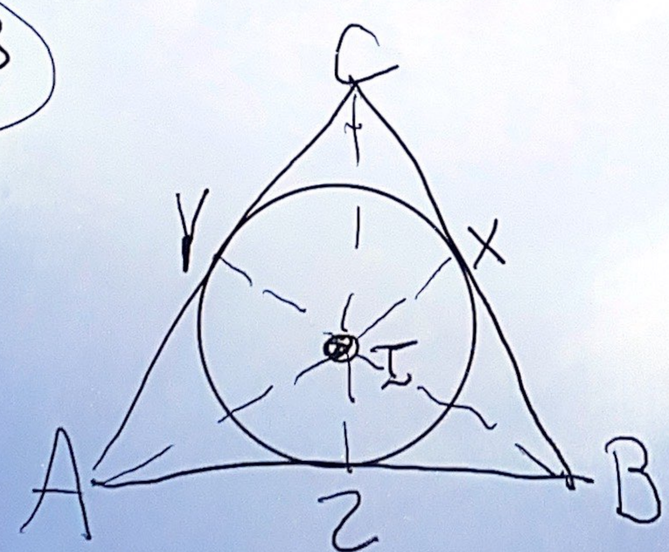
CI is an angle bisector of C (YI ≅ XI)

The point of intersection of angle bisectors is the incenter

The angle bisectors are concurrent since

$XI = YI = ZI$

4.2.3



Inradius i the same

so $IY = IX = IZ = r$

$\Delta AYI \cong \Delta AZI$ so $YI \cong IZ$

($\angle YAI \cong \angle ZAI$, AI common as

Similarly $\Delta IYC \simeq \Delta IXC$ so
$YI \cong IX = r$

Thus a circle with center I and radius r is tangent to each side

3 angle bisectors are concurrent (all three intersect in one point though two are needed for the proof

All the radii are equal
so r is the inradius

Therefore incenter exists for any Δs and is inside the Δs

point of inters on e is equidistant from 3 sides

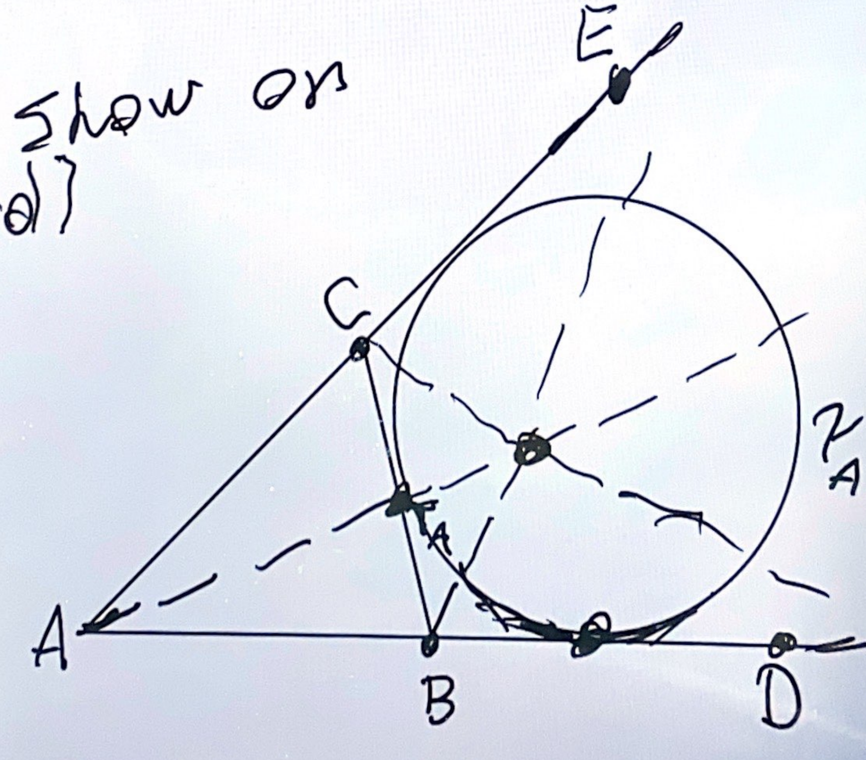
Def: The point of concurrency of the bisectors is called the incenter of the triangle. The distance from the incenter to the sides of the triangle is called the inscribed circle of the triangle

4.3 The Escribed and the excenters

Definition: A circle that is outside the triangle and is tangent to all three side-lines of the triangle. A circle that is tangent to all three escribed circle or excircle. The center of an excircle is called an excenter.

Note there are actually 3 excenters though I will only show one here you will draw all of them for hw

(Can show on board)



This is the excenter opposite A or A-excircle and written as γ_A

written as γ/A

Hint for 4.3.4: Look for congruent triangles

Note on 4.5 - Heron's formula

r

There is both an incenter and excente here. So there are a lot of moving parts here

H.W

#4.1.5, 4.2.1, 4.2.2, 4.3.3, 4.3.4