Practice Exam Solutions

1) $y' = (1-2x)y^2$; $y(0) = \frac{-1}{12}$

$$\downarrow$$

$$y'y^{-2} = 1-2x$$

$$\int \frac{dy}{\cancel{dx}} y^{-2} \cancel{dx} = \int 1-2x\,dx$$

$$\int y^{-2}dy = \int 1\,dx - \int 2x\,dx$$

$$y^{-1} = x - x^2 + C$$

$$y = \frac{1}{x - x^2 + C}$$

$$y(0) = \frac{-1}{12} = \frac{1}{(0)-0^2+C}$$

$$\frac{-1}{12} = \frac{1}{C}$$

$$-12 = C \qquad \boxed{y = \frac{1}{x-x^2-12}}$$

2) $3x^2 + 9 - 2xy + (-x^2 + 24y^2 + 1)y' = 0$

Exact Equation

$M_y = -2x$ ✓

$N_x = -2x$ ✓

Can solve as exact

$F(x,y) = \int 3x^2 + 9 - 2xy\, dx$

$= \int 3x^2 dx + \int 9 dx - \int 2xy\, dx$

$= x^3 + 9x - x^2 y + \phi y$

$F_y(x,y) = \phi' y - x^2 = -x^2 + 24y^2 + 1$

$\int \phi' y\, dy = \int 24y^2 dy + \int 1 dy$

$\phi y = 8y^3 + y$

$0 = 8y^3 + y + c$

$\boxed{F(x,y) = x^3 + 9x - x^2 y + 8y^3 + y + c}$

3) $\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 16y = x$

char. poly: $r^2 + 8r + 16 = 0$

$(r+4)^2 = 0$

$r = -4$

$y_1 = ve^{-4x}$

$y_1' = -4ve^{-4x} + v'e^{-4x}$

$$= e^{-4x}(-4u + u')$$

$$y_1'' = -4e^{-4x}(-4u + u') + e^{-4x}(-4u' + u'')$$

$$= e^{-4x}(u'' - 8u' - 16u)$$

$$e^{-4x}(u'' - 8u' + 16u) + 8e^{-4x}(-4u + u') + 16(ue^{-4x}) = x$$

$$e^{-4x}(u'' - \cancel{8u'} + \cancel{16u} - \cancel{34u} + \cancel{8u'} + \cancel{16u}) = x$$

$$e^{-4x}u'' = x$$

$$u'' = xe^{4x}$$

$$\int u''dx = \int xe^{4x}dx \qquad \begin{matrix} u = x & dv = e^{4x} \\ du = 1 & v = \frac{1}{4}e^{4x} \end{matrix}$$

$$u' = \frac{1}{4}xe^{4x} - \frac{1}{4}\int e^{4x}dx$$

$$u' = \frac{1}{4}xe^{4x} - \frac{1}{16}e^{4x} + a$$

$$\int u'dx = \frac{1}{4}\int xe^{4x}dx - \frac{1}{16}\int e^{4x}dx + \int a\,dx$$

$$u = \frac{1}{16}xe^{4x} - \frac{1}{64}e^{4x} - \frac{1}{64}e^{4x} + ax + b$$

$$u = e^{4x}\left(\frac{1}{16}x - \frac{1}{32}\right) + ax + b$$

$$y = \left(\cancel{e^{4x}}\left(\frac{1}{16}x - \frac{1}{32}\right) + ax + b\right) \cdot \cancel{e^{-4x}}$$

$$\boxed{y = \frac{1}{16}x - \frac{1}{32} + (ax + b)e^{-4x}}$$

4) $(x+6)y'' - (9-x)y' + y = 0$

interval of convergence:

$(-6, 6)$

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=0}^{\infty} n(n-1) x^{n-2}$$

$$(x+6)\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} - (9-x)\sum_{n=0}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) x^{n-1} + \sum_{n=2}^{\infty} 6n(n-1) a_n x^{n-2} - \sum_{n=1}^{\infty} 9 n a_n x^{n-1} + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} 6(n+1)(n+2) a_{n+2} x^n - \sum_{n=0}^{\infty} 9(n+1) a_{n+1} x^n - \sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[6(n+1)(n+2) a_{n+2} + (n-9)(n+1) a_{n+1} - (n+1) a_n \right] x^n$$

$$6(n+1)(n+2) a_{n+2} + (n-9)(n+1) a_{n+1} - (n+1) a_n = 0$$

$$6(n+1)(n+2) a_{n+2} = (n+1) a_n - (n-9)(n+1) a_{n+1}$$

$$a_{n+2} = \frac{(\cancel{n+1}) a_n - (n-9)(\cancel{n+1}) a_{n+1}}{6(\cancel{n+1})(n+2)}$$

$$a_{n+2} = \frac{(9-n) a_{n+1} - a_n}{6(n+2)} \qquad n \geq 1$$

$$n=0 \quad a_2 = \frac{(9-0)a_1 - a_0}{6(0+2)} = \frac{9}{12}a_1 - \frac{1}{12}a_0$$

$$= \frac{3}{4}a_1 - \frac{1}{12}a_0$$

$$n=1 \quad a_3 = \frac{(9-1)a_2 - a_1}{6(1+2)} = \frac{8}{18}a_2 - \frac{1}{18}a_1$$

$$= \frac{4}{9}a_2 - \frac{1}{18}a_1$$

$$= \frac{1}{3}a_1 - \frac{1}{27}a_0 - \frac{1}{18}a_1$$

$$= \frac{5}{18}a_1 - \frac{1}{27}a_0$$

$$n=2 \quad a_4 = \frac{(9-2)a_3 - a_2}{6(2+2)} = \frac{7}{24}a_3 - \frac{1}{24}a_2$$

$$= \frac{35}{432}a_1 - \frac{7}{648}a_0 - \frac{3}{32}a_1 - \frac{1}{288}a_0$$

$$= \frac{-11}{864}a_1 - \frac{37}{2592}a_0$$

$$\boxed{y = a_1\left[x + \frac{3}{4}x^2 + \frac{5}{18}x^3 - \frac{11}{864}x^4 + \ldots\right] + a_0\left[1 - \frac{1}{18}x^2 - \frac{1}{27}x^3 - \frac{37}{2592}x^4 + \ldots\right]}$$

5) $y'' + 16y = 32t$; $y(0) = 3$, $y'(0) = 7$

$$s^2 Y(s) - s\,y(0) - y'(0) + 16 = \frac{32}{s^2}$$

$$Y(s)(s^2+16) - 3s - 7 = \frac{32}{s^2}$$

$$Y(s) = \frac{3s^3 + 7s^2 + 32}{s^2(s^2+16)}$$

$$\frac{3s^3 + 7s^2 + 32}{s^2(s^2+16)} = \frac{As+B}{s^2} + \frac{Cs+D}{s^2+16}$$

$$3s^3 + 7s^2 + 32 = As + B(s^2+16) + Cs + D(s^2)$$

$$= As^3 + 16As + Bs^2 + 16B + Cs^3 + Ds^2$$

$$= s^3(A+C) + s^2(B+D) + s(16A) + (16B)$$

$3 = A + C$ $7 = B + D$ $0 = 16A$ $32 = 16B$

$C = 3$ $D = 5$ $A = 0$ $B = 2$

$$Y(s) = \frac{2}{s^2} + \frac{3s+5}{s^2+16}$$

$$= \frac{2}{s^2} + \frac{3}{s} + \frac{5}{s^2+16}$$

$$\mathcal{L}^{-1} Y(s) = 2\mathcal{L}\left(\frac{1}{s^2}\right) + 3\mathcal{L}^{-1}\left(\frac{1}{s}\right) + \frac{4}{5}\left(\frac{4}{s^2+16}\right)$$

$$\boxed{y(t) = 2t + 3 + \frac{4}{5}\sin(4t)}$$