Test # 4 Review

Textbook question: 7.3, #3, pg. 338.

☆ $(1-2x^2)y'' + (2-6x)y' - 2y = 0,\ y(0)=1,\ y'(0)=0$

Let $y = \sum_{n=0}^{\infty} a_n x^n$

$y' = \sum_{n=0}^{\infty} (n) a_n x^{n-1}$

$y'' = \sum_{n=0}^{\infty} (n-1)(n) a_n x^{n-2}$

Plug into ☆

$$LS = \sum_{n=0}^{\infty}(n-1)(n)a_nx^{n-2} - \sum_{n=0}^{\infty}2(n-1)(n)a_nx^{n-2} + \sum_{n=0}^{\infty}2(n)a_nx^{n-1} - \sum_{n=0}^{\infty}6(n)a_nx^{n-1} - \sum_{n=0}^{\infty}2a_nx^n$$

↓peel ↓peel ↓peel ↓peel

$$= \sum_{n=2}^{\infty}(n-1)(n)a_nx^{n-2} - \sum_{n=2}^{\infty}2(n-1)(n)a_nx^{n-2} + \sum_{n=1}^{\infty}2(n)a_nx^{n-1} - \sum_{n=1}^{\infty}6(n)a_nx^{n-1} - \sum_{n=0}^{\infty}2a_nx^n$$

↓shift 2 ↓shift 2 ↓shift 1 ↓shift 1

$$= \sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n - \sum_{n=0}^{\infty}2(n+1)(n+2)a_{n+2}x^n + \sum_{n=0}^{\infty}2(n+1)a_{n+1}x^n - \sum_{n=0}^{\infty}6(n+1)a_{n+1}x^n - \sum_{n=0}^{\infty}2a_nx^n$$

$$LS = \sum_{n=0}^{\infty}\left[(n+1)(n+2)a_{n+2} - 2(n+1)(n+2)a_{n+2} + 2(n+1)a_{n+1} - 6(n+1)a_{n+1} - 2a_n\right]x^n$$

$$(n+1)(n+2)a_{n+2} - 2(n+1)(n+2)a_{n+2} + 2(n+1)a_{n+1} - 6(n+1)a_{n+1} - 2a_n = 0$$

$$\left((n+1)(n+2) - 2(n+1)(n+2)\right)a_{n+2} = -2(n+1)a_{n+1} + 6(n+1)a_{n+1} + 2a_n$$

$$a_{n+2} = \frac{-2(n+1)a_{n+1} + 6(n+1)a_{n+1} + 2a_n}{(n+1)(n+2) - 2(n+1)(n+2)}$$

$n=0$: $$a_2 = \frac{-2(1)a_1 + 6(1)a_1 + 2a_0}{(1)(2) - 2(1)(2)} = -\frac{4a_1 + 2a_0}{2} = -(2a_1 + a_0) = \boxed{-2a_1 - a_0}$$

$n=1$: $$a_3 = \frac{-2(2)a_2 + 6(2)a_2 + 2a_1}{(2)(3) - 2(2)(3)} = \frac{-4a_2 + 12a_2 + 2a_1}{6-12} = -\frac{8a_2 + 2a_1}{6} = -\frac{8}{6}a_2 - \frac{2}{6}a_1$$

$$= -\frac{8}{6}(-2a_1 - a_0) - \frac{2}{6}a_1 = \frac{16}{6}a_1 + \frac{8}{6}a_0 - \frac{2}{6}a_1 = \boxed{\frac{14}{6}a_1 + \frac{8}{6}a_0}$$

$n=2$: $$a_4 = \frac{-2(3)a_3 + 6(3)a_3 + 2a_2}{(3)(4) - 2(3)(4)} = \frac{12a_3 + 2a_2}{-12} = -a_3 - \frac{1}{6}a_2 = -\frac{14}{6}a_1 - \frac{8}{6}a_0 + \frac{2}{6}a_1 + \frac{1}{6}a_0$$

$$= \frac{-12}{6}a_1 - \frac{7}{6}a_0 = \boxed{-2a_1 - \frac{7}{6}a_0}$$

$n=3$ | $a_5 = \dfrac{-2(4)a_4 + 6(4)a_4 + 2a_3}{(4)(5) - 2(4)(5)} = \dfrac{-8a_4 + 24a_4 + 2a_3}{20-40} = \dfrac{16a_4 + 2a_3}{-20}$

$= -\frac{16}{20}\left(-2a_1 - \frac{7}{6}a_0\right) - \frac{1}{10}\left(\frac{14}{6}a_1 + \frac{8}{6}a_0\right)$

$= \frac{+32}{20}a_1 + \frac{14}{15}a_0 - \frac{14}{60}a_1 - \frac{8}{60}a_0$

$= \frac{41}{30}a_1 + \frac{4}{5}a_0$

So the series solution of ✡ is

$$a_0 + a_1x + (-2a_1 - a_0)x^2 + \left(\tfrac{7}{3}a_1 + \tfrac{4}{3}a_0\right)x^3 + \left(-2a_1 - \tfrac{7}{6}a_0\right)x^4 + \left(\tfrac{41}{30}a_1 + \tfrac{4}{5}a_0\right)x^5 + \cdots$$

$$= a_0\left(1 - x^2 + \tfrac{4}{3}x^3 - \tfrac{7}{6}x^4 + \cdots\right) + a_1\left(x - 2x^2 + \tfrac{7}{3}x^3 - 2x^4 + \tfrac{41}{30}a_1 + \cdots\right)$$

Evaluate IVP: $y(0)=1, y'(0)=0$

$y = \sum_{n=0}^{\infty} a_n x^n$

$1 = \sum_{n=0}^{\infty} a_n (0)^n = a_0$

$\boxed{a_0 = 1}$

$y' = \sum_{n=0}^{\infty} (n) a_n x^{n-1}$

$0 = \sum_{n=0}^{\infty} (n) a_n (0)^{n-1} = a_1$

$a_1 = 0$

So, plugging in a_0 and a_1 gives

$= 1 - x^2 + \frac{4}{3}x^3 - \frac{7}{6}x^4$

oops, this is not the right answer!

Some arithmetic error somewhere...