Series Solutions of Linear Systems

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Power Series

To best fit a function, we can take higher degree polynomials called Taylor Polynomials. Polynomials of infinite degree are called Power Series. The Taylor Series of y(x) centered at x=a is given by: $T(x) = \sum_{n=0}^{\infty} \frac{y^{(n)}(a)}{n!}(x-a)^n$.

Note: Maclaurin series is Taylor Series where x = 0.

So $y(x) = T(x)$ **in the interval of convergence.**

Part of a power series calculation:

Let $y = \sum_{n=0}^{\infty} a_n x^n$ be a series representation of the function $y = y(x)$ on an interval I containing 0 .

(a) Express $(\star)(2-x)y'' + 2y$ as a power series on the interval I.

Algorithm:

1. Take the derivatives of y and plug them into $(\star)$.
2. Then take each sigma and perform a **shift** to a common term (ex: $n = 0$).
3. Write as a single summation: $\sum_{n=0}^{\infty}[2(n+2)(n+2)a_{n+2} - (n+1)(n)a_{n+1} + 2a_n]x^n$

(b) Find the coefficients for $(\star)$ when $y = \sum_{n=0}^{\infty} a_n x^n$.

Algorithm:

1. Set inside of the single summation from (a) and set to 0: $2(n+2)(n+2)a_{n+2} - (n+1)(n)a_{n+1} + 2a_n = 0$.
2. Solve for a_{n+2}. This is our **recurrence relation**.
3. Now you can calculate n=0, n=1, etc to get the coefficients in terms of c_0, c_1.
4. Write it out: The series rep. of the solution $y = \sum_{n=0}^{\infty} a_n x^n$ is:
   $= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ...$
   $= a_0 + a_1 x + (-\frac{1}{2}a_0)x^2 + (-\frac{1}{12}a_0 - \frac{1}{6}a_1)x^3 ...$
   $= a_0(1 - \frac{1}{2}x^2 - \frac{1}{12}x^3 + ...) + a_1(x - \frac{1}{6}x^3)$
   This is the series representation of $(\star)$ (a_0, a_1 are free parameters.)

(c) Find the series representation of the IVP $(\star)$ with $y(0) = 1, y'(0) = -1$.

Algorithm:

1. Plug IVP into y, y', y'' to get your a_0, a_1.
2. Plug these into your equations from above.

Ordinary Point
$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0$.
Def: x_0 is an ordinary point is $P_0(x_0) \neq 0$. Otherwise, singular point.

Example: $(x^2+1)y'' - (7-x)y' + y = 0$. Find interval for series soln.
$P_0(x) = x^2 + 1 = 0$
$x^2 = -1$
$x = \pm i$
$\rho = |0 - i| = 1$ <- note, magnitude, not absolute value
$\rho = |0 - (-i)| = 1$

Interval: $(0-1, 0+1) = (-1, 1)$
Magnitude of complex number $a + bi$: $\sqrt{a^2 + b^2}$.