Linear Second Order Equations Formula Sheet

Linear Second Order Equations

$y'' + p(x)y' + q(x)y = f(x)$

Constant Coefficient Homogenous Equations

For everything, remember to check linear independence: $\frac{y_1}{y_2}$ is not constant.

Different Roots

$y'' - 5y' + 4y = 0$

Characteristic Polynomial: $r^2 - 5r + 4 = (r-4)(r-1)$. Roots = 1,4.

Basic Soln: $y_1 = e^{r_1 x}, y_2 = e^{r_2 x}$

Repeated Roots

$y'' + 18y' + 81y = 0$

Characteristic Polynomial: $r^2 + 18r + 81 = (r+9)(r+9)$. Roots = -9,-9.

Basic Soln: $y_1 = e^{r_1 x}, y_2 = xe^{r_1 x}$

Complex Roots

$y'' + 16y' + 145y = 0$

Roots: $-8 + 9i, -8 - 9i$, Format: $\lambda \pm \omega i$

Basic Soln: $y_1 = e^{\lambda t} sin(\omega t), y_2 = e^{\lambda t} cos(\omega t)$

Second Order Nonhomogenuous

$y'' - 5y' + 4y = 22 - 8t$

First, solve the associated homogenous eqn: $y'' - 5y' + 4y = 0$ to get your y_1, y_2.

Let y_p be a *particular solution* of the nonhomogenous solution.

Then look at the RS and try $y_p = Ax + B$.

So, $y_p' = A$ and $y_p'' = 0$.

Then plug these in to $y'' - 5y' + 4y = 22 - 8t$ and solve for A and B.

The general solution is $y(t) = y_p + c_1 y_1 + c_2 y_2$.

Principle of Superposition

Let y_{p_1} be particular soln of $y'' + p(x)y' + q(x)y = f_1(x)$

Let y_{p_2} be particular soln of $y'' + p(x)y' + q(x)y = f_2(x)$

Then $y_{p_1} + y_{p_2}$ is a soln of $y'' + p(x)y' + q(x)y = f_1(x) + f_2(x)$

Method of Undetermined Coefficients

$y'' - 4y' + 3y = e^{3x}(6 + 8x + 12x^2)$. Note if clause in RS parans is constant, just do $y_p = Ae^{3x}$

1. Write $y_p = ue^{3x}$, find y', y'':
   $y' = u'e^{3x} + 3ue^{3x}$
   $y'' = u''e^{3x} + 3u'e^{3x} + 3u'e^{3x} + 9ue^{3x} = e^{3x}(u'' + 6u' + 9u)$
   a. **Shortcut:** For $y = ue^{\alpha x}, y'' = e^{\alpha x}(u'' + 2\alpha u' + \alpha^2 u)$
2. Plug it in to LS. Results in $e^{3x}(u'' + 2u') = e^{3x}(12x^2 + 8x + 6)$
3. Try particular soln $u_p = Ax^2 + Bx + C$ with derivatives $u' = 2Ax + b, u'' = 2A$
4. Plug in to left side, so $(4A)x + (2A + 2B) = 12x^2 + 8x + 6$
   1. This is no good! Doesn't match exponents of polynomial!
5. Instead, try higher power $u_p = Ax^3 + Bx^2 + Cx + D$, plug in, and solve for u_p. Once you have it, remember to plug in to $y = u * e^{3x}$.

Reduction Of Order

$xy'' - (2x+1)y' + (x+1)y = x^2$. Given $y_1 = e^x$ for associated homog eqn.

- Find general solution for nonhomog equation.
  1. Plug y_1 into var. of params:
     - $y = u * y_1$
     - $y' = u'y_1 + uy_1$ (diff)
     - $y'' = ...$ ((diff again)
     - Plug these into the nonhomog eqn.
       - End up with something like $e^x(xu'' - u')$. To make it first order, let $u' = z$.
       - Rearrange resulting eqn into recognizable form: $z' - \frac{1}{x}z = xe^{-x}$
       - Solve using Chapter 2 techniques. Seperable Eqn or var of params.
       - Now you have z. We want u, our mystery function. $z = u'$, so integrate z.
       - $u = \int zdx$
     - Once you have u, plug it in to $y = u * y_1$. **This is our general solution for our nonhomog. eqn.**
     - Find y_p by setting c_1 and c_2 to something simple like 0.
     - We know $y1$, it was given.
     - Solve for $y2$ in $y = y_p + c_1y_1 + c_2y_2$
       - Find a y_{p_2} by setting c_1 and c_2 to something else.
       - Then $y_2 = y_{p_1} - y_{p_2}$
       - Plug it all in!

Variation of Parameters (2nd Order Nonhomog)

$x^2y'' - 2xy' + 2y = x^{9/2}$. Given $y_1 = x, y_x = x^2$ form a fundamental set of solns of assoc. homog. eqn. Find particular and general soln.

1. Toolbox!
    1. $y_p = u_1 y_1 + u_2 y_2$
    2. Assume $u_1' y_1 + u_2' y_2 = 0$
2. Take $y_p = u_1 y_1 + u_2 y_2 = u_1 x + u_2 x^2$ and find y', y''. Use $u_1' y_1 + u_2' y_2 = 0$ to simplify.
3. Plug y, y', y'' into the nonhomog eqn, simplify.
    1. Once simplified, combine this simplified equation with our assumption $u_1' y_1 + u_2' y_2 = 0$.
    2. This is now a 2x2 linear system with vars u_1', u_2'. Solve.
4. Once solved, stick into $y_p = u_1 y_1 + u_2 y_2$. Let constants be 0. This is our particular solution.
5. General solution given by $y = y_p + c_1 y_1 + c_2 y_2$ as always.