MAT 2680 Differential Equations Project 2

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1 Section 6.1 Problem 11

A unit mass hangs in equilibrium from a spring with a constant $k = \frac{1}{16}$. Starting at $t = 0$, a force $F(t) = 3\sin(t)$ is applied to the mass. Find its displacement for $t > 0$.

To start out, we need to derive the Differential Equation we'll be working with for this problem. The textbook gives us a general equation of motion for mass:

$$my'' + cy' + ky = F(t)$$

The problem gives us a few given values: m, k, and $F(t)$. Since we're working with a unit mass, the mass in our problem will be $1g$. Section 6.1 focuses specifically on working with undampened springs, meaning our value of c will be 0. Lastly, we're provided $F(t) = 3\sin t$. Plugging our given values into the equation of motion for mass, we get:

$$y'' + \tfrac{1}{16}y = 3\sin(t)$$

Now, we have a second-order linear non-homogeneous equation! Let's go through what finding the solution for this equation looks like.

First, we need to find the solution of the associated homogeneous equation:

$$\begin{aligned} y'' + \frac{1}{16}y &= 0 \\ r^2 + \frac{1}{16} &= 0 \\ r^2 &= \frac{-1}{16} \\ r &= \frac{1}{4}i \end{aligned}$$

Since this is a case 3 homogeneous second-order equation, the solution to y_h will be:

$$y_h = c_1 \sin \tfrac{1}{4}t + c_2 \cos \tfrac{1}{4}t.$$

Now that we have our y_h, we now need to find our particular solution y_p. To do this, let's start off by assuming that y_p will take the form of $A \sin t$. Then,

$$\begin{aligned} \mathrm{y}_p &= A \sin t \\ \mathrm{y}'_p &= A \cos t \\ \mathrm{y}''_p &= -A \sin t \end{aligned}$$

Plugging back into our original equation:

$$\begin{aligned} -A \sin t - \tfrac{1}{16}(A \sin t) &= 3 \sin t \\ \tfrac{-15}{16} A \sin t &= 3 \sin t \\ \tfrac{-15}{16} A &= 3 \\ A &= \tfrac{-16}{15} * 3 \\ &= \tfrac{-48}{15} \\ &= \tfrac{-16}{5} \end{aligned}$$

Bringing our y_h and y_p together, we get the general solution of:

$$y(t) = \frac{-16}{5} \sin t + c_1 \sin \frac{1}{4} t + c_2 \sin \frac{1}{4} t$$

This general solution represents the displacement or change in position acting on our mass at some time t. However, we know from the problem that at $t = 0$, our mass is hanging in equilibrium. When something is in equilibrium, that implies that its positionality, or $y(0)$ will be equal to 0. Therefore, $y'(0) = 0$ as well. Remember that for initial value problems, we're typically given values for $y(t)$ and $y'(t)$.

$$\begin{aligned} y(0) &= \tfrac{-16}{5} \sin(0) + c_1 \sin(0) + c_2 \cos(0) \\ y'(0) &= \tfrac{-16}{5} \cos(0) + \tfrac{1}{4} c_1 \cos(0) - \tfrac{1}{4} c_2 \sin(0) \\ 0 &= c_2 \\ 0 &= \tfrac{-16}{5} + \tfrac{1}{4} c_1 \\ \tfrac{16}{5} &= \tfrac{1}{4} c_1 \\ \tfrac{64}{5} &= c_1 \end{aligned}$$

Therefore, the particular solution for the displacement of our mass on an undampened spring at $t > 0$ can be found with the equation:

$$y(t) = \frac{-16}{5}\sin(t) + \frac{64}{5}\sin\frac{1}{4}(t)$$