A 4 pound object is suspended from a spring. The spring stretches an extra 6 inches with the weight attached. If the spring is stretched by 3 inches and released with a downward velocity of 40 feet per second, find a formula describing the displacement of the object using only a single cosine function.

First we gather our given values:

weight: 4 pounds

Spring Elongation (s): 6 inches or 6/12 = 1/2 ft

Initial Displacement (x, u(0)): 3 inches or 3/12 feet or 1/4 ft.

Acceleration due to the force of gravity (g): 32.17 ft/s^2 we will use 32 ft/s^2

y'(0)= - 40 ft/s (initial downward velocity, derivative of the initial position)

knowing that:

$$mg = ks$$

$$\frac{g}{s} = \frac{k}{m}$$

We get:

$$\frac{32}{1/2} = \frac{k}{m} = 64$$

Plugging into equation (6.1.3) we get:

$$u'' + 64u = 0 \tag{6.1.4}$$

The characteristic equation for (6.1.4) in the form of:

$$r^2 + br + c = 0$$

Values plugged in:

$$r^2 + 0r + 64 = 0$$

Finally:

$$r^2 + 64 = 0$$

$$r^2 + 64 = 0$$

We now solve the 2nd order linear differential equation by identifying the roots:

$$r^2 = -64$$
$$r = \sqrt{-64} = 8i$$

Therefore the general solution of (6.1.4) is:

$$u(t) = c_1 cos(8t) + c_2 sin(8t) \tag{6.1.5}$$

Apply the initial condition:

$$u(0) = -1/4$$

$$c_1 cos(0) + c_2 sin(0) = -1/4$$

$$c_1 = -\frac{1}{4}$$

Differentiating (6.1.5) yields:

$$u'(t) = -8\, c_1\, sin(8t) + 8\, c_2\, cos(8t) \tag{6.1.6}$$

Apply the initial condition:

$$u'(0) = 40$$

$$u'(0) = -8\, c_1\, sin(0) + 8\, c_2\, cos(0) = 40$$

$$u'(0) = 8\, c_2 = 40$$

$$c_2 = 5$$

Therefore:

$$u(t) = -\frac{1}{4} cos(8t) + 5 sin(8t)$$

Adjusting form to:

$$R = \sqrt{c_1^2 + c_2^2}$$

$$c_1 = R\, cos\ \phi$$

$$c_2 = R\, sin\ \phi$$

$$cos(\phi) = \frac{c_1}{R}$$

$$sin(\phi) = \frac{c_2}{R}$$

$$R = \sqrt{-\frac{1}{4}^2 + 5^2} = 5.00625$$

$$\text{(eq. a)} \quad cos(\phi) = \frac{-\frac{1}{4}}{5.006} = \ 0.0499$$

$$\text{(eq. b)} \quad sin(\phi) = \frac{5}{5.006} = 0.9988$$

Using a calculator:

$$\text{(eq. a)}\ \ \phi = 1.621 \text{ rad}$$

$$\text{(eq. b)}\ \ \phi = 1.522 \text{ rad}$$

Finally in form:

$$u = \ R\, cos(\omega t - \phi)$$

$$u(t) = \ 5.006\, cos(8t - 1.57)$$