Reduction of order 5.6 problem 3 (textbook)

$x^2y'' - xy' + y = x \quad y_1 = x$

$\boxed{y = uy_1}$

$= ux$

$y' = u + u'x$

$y'' = u' + u' + u''x = 2u' + u''x$

$x^2(2u' + u''x) - x(u + u'x) + ux$

$2u'x^2 + u''x^3 - \cancel{x}u - x^2u' + ux$

$u''x^3 + u'x^2 = x$

$x^2(u''x + u') = x$

$\frac{xu'' + u'}{x} = \frac{\frac{1}{x}}{x}$ $\quad u'' = z'$

$u'' + \frac{1}{x}u' = \frac{1}{x^2}$ $\quad \boxed{u' = z}$

$z' + \frac{1}{x}z = \frac{1}{x^2}$

$z' + \frac{1}{x}z = \frac{1}{x^2}$

$z' + \frac{1}{x}z = 0$

$\frac{z'}{z} = -\frac{1}{x}$

$\int \frac{\frac{dz}{dx}}{z}\,dx = \int -\frac{1}{x}$

$\ln|z| = -\ln(x) + K$

$|z| = -|x| \cdot C$

$C = 1$

Let $z_1 = \frac{1}{x}$

$\boxed{z = v z_1} = v \cdot \frac{1}{x}$

$z' = v \cdot -\frac{1}{x^2} + v' \cdot \frac{1}{x}$

$z' + \frac{1}{x}z$

$-\frac{v}{x^2} + \frac{v'}{x} + \frac{1}{x} \cdot \frac{v}{x}$

$\frac{1}{x}v'$

$\frac{1}{x}v' = \frac{1}{x^2}$

$v' = \frac{1}{x}$

$v = \ln x + c_2$

Back substitute:

$z = v \cdot \frac{1}{x}$

$z = \ln x + c_2 \cdot \frac{1}{x}$

$z = \frac{\ln x + c_2}{x}$

$u' = z$

$u' = \int \frac{\ln x + c_2}{x}\,dx$

$u = \ln x + c_2$

$du = \frac{1}{x}\,dx$

$dx = x\,du$

$= \int u\,du$

$= \frac{u^2}{2}$

$= \frac{(\ln(x) + c_2)^2}{2} + c_1$

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$$= \frac{(\ln(x) + C_2)^2}{2} + C_1$$

$y = u \cdot x$

$$= \left(\frac{(\ln(x) + C_2)^2}{2} + C_1 \right) x$$

My answer:

$$= \frac{x(\ln(x) + C_2)^2}{2} + xC_1$$