

5.7 #1

We are given

$$y'' + 9y = \tan(3x)$$

set the homogeneous eq.

$$y'' + 9y = 0$$

solve for the characteristic

$$\text{eq. } r^2 + 9 = 0$$

$$r = \pm\sqrt{-9}$$

$$r = \pm 3i$$

Homogeneous eq.:

$$Y_h = A\cos(3x) + B\sin(3x)$$

Wronskian of $y_1 = \cos(3x)$
 $y_2 = \sin(3x)$

$$W(y_1, y_2) = \begin{vmatrix} \cos(3x) & \sin(3x) \\ -3\sin(3x) & 3\cos(3x) \end{vmatrix}$$

$$W = [\cos(3x) \cdot 3\cos(3x) - \sin(3x) \cdot (-3\sin(3x))]$$

$$W = 3[\cos(3x)]^2 + 3[\sin(3x)]^2$$

$$W = 3[(\cos(3x))^2 + (\sin(3x))^2]$$

$$W = 3 \cdot 1$$

$$W(y_1, y_2) = 3$$

We know that there is a set of solutions for every value of x for y_1 and y_2 on the interval $(0, \infty)$

so,

$$y(x) = -Y_h(x) \int \frac{y_2(x) g(x)}{W(y_1, y_2)(x)} dx + Y_g(x) \int \frac{y_1(x) g(x)}{W(y_1, y_2)(x)} dx$$

$$\begin{aligned} &= -\cos(3x) \int \frac{\sin(3x) \cdot \tan(3x)}{3} dx + \sin(3x) \int \frac{\cos(3x) \cdot \tan(3x)}{3} dx \\ &= -\cos(3x) \int \frac{\sin(3x) \cdot \frac{\sin(3x)}{\cos(3x)}}{3} dx + \sin(3x) \int \frac{\cos(3x) \cdot \frac{\sin(3x)}{\cos(3x)}}{3} dx \\ &= -\frac{\cos(3x)}{3} \int \frac{(\sin(3x))^2}{\cos(3x)} dx + \frac{\sin(3x)}{3} \int \sin(3x) dx \\ &= -\frac{\cos(3x)}{3} \int \frac{1 - (\cos(3x))^2}{\cos(3x)} dx + \frac{\sin(3x)}{3} \int \frac{-\cos(3x)}{\cos(3x)} dx \\ &= -\frac{\cos(3x)}{3} \int \left(\frac{1}{\cos(3x)} - \cos(3x) \right) dx + \frac{\sin(3x)}{3} \int \left(\frac{-\cos(3x)}{\cos(3x)} \right) dx \\ &= -\frac{\cos(3x)}{3} \int \left(\frac{1}{\cos(3x)} - \cos(3x) \right) dx + \frac{\sin(3x)}{3} \int \left(\frac{-\cos(3x)}{\cos(3x)} \right) dx \\ &= -\frac{\cos(3x)}{3} \int \sec(3x) dx - \frac{\cos(3x)}{3} \int \cos(3x) dx - \frac{\sin(3x)}{3} \int \frac{1}{\cos(3x)} dx \end{aligned}$$

$$\frac{1}{3} \ln|\sec(3x) + \tan(3x)| - \frac{\sin(3x)}{3}$$

Going back!

$$\begin{aligned} &= -\frac{1}{9} \cos(3x) \cdot \ln|\sec(3x) + \tan(3x)| \\ &\quad + \frac{\sin(3x) \cos(3x)}{9} - \frac{\sin(3x) \cos(3x)}{9} \end{aligned}$$

$$Y(x) = -\frac{1}{9} \cos(3x) \cdot \ln|\sec(3x) + \tan(3x)|$$

Thus the general solution is:

$$y = Y_h + Y(x)$$

$$\begin{aligned} &y = A\cos(3x) + B\sin(3x) - \frac{1}{9} \cos(3x) \cdot \ln|\sec(3x) + \tan(3x)| \end{aligned}$$