Section 5.7 Variation of Parameters Problem 1

$y'' + 9y = \tan(3x)$

$r^2 + 9 = 0$

$\sqrt{r^2} = \sqrt{-9}$

$r = \pm 3i$

$y = c_1 \cos(3x) + c_2 \sin(3x)$

$y_p = u_1 y_1 + u_2 y_2$

$y_1 = \cos(3x)$ $\quad$ $y_2 = \sin(3x)$ $\quad$ $g(x) = \tan(3x)$

$y_1' = -3\sin(3x)$ $\quad$ $y_2' = 3\cos(3x)$ $\quad$ $w(y_1, y_2) = y_1 y_2' - y_1' y_2$

$$u_1 = \int -\frac{y_2 g(x)}{w(y_1, y_2)}$$

$$u_2 = \int \frac{y_1 g(x)}{w(y_1, y_2)}$$

$$\begin{aligned} w(y_1, y_2) &= (\cos(3x))(3\cos(3x)) - (-3\sin(3x))(\sin(3x)) \\ &= 3\cos^2(3x) + 3\sin^2(3x) \\ &= 3(\cos^2(3x) + \sin^2(3x)) \\ &= 3(1) \\ &= 3 \end{aligned}$$

$$u_1 = \int -\frac{\sin(3x)\tan(3x)}{3}\,dx \qquad \sec(x) = \frac{1}{\cos(x)} \quad u = \tan(3x) \quad du = 3\sec^2(3x)dx$$

$$v = -\frac{1}{3}\cos(3x) \quad dv = \sin(3x)dx$$

$$= -\frac{1}{3}\int \sin(3x)\tan(3x)\,dx$$

$$\int uv = uv - \int v\,du$$

$$= -\frac{1}{3}\left(\left(-\frac{1}{3}\cos(3x)\right)(\tan(3x)) - \int \left(-\frac{1}{3}\cos(3x)\right)(3\sec^2(3x))dx\right.$$

$$= -\frac{1}{3}\left(\left(-\frac{1}{3}\cos(3x)\right)(\tan(3x)) - \left((3)\left(-\frac{1}{3}\right)\right)\int \cos(3x)\sec^2(3x)\,dx\right)$$

$$= -\frac{1}{3}\left(\left(-\frac{1}{3}\cos(3x)\right)(\tan(3x)) - \left(-\int \sec(3x)dx\right)\right) \qquad \boxed{u = 3x \quad du = 3dx \quad dx = \frac{1}{3}du}$$

$$= -\frac{1}{3}\left(\left(-\frac{1}{3}\cos(3x)\right)(\tan(3x)) - \left(-\int \sec(u)\,\frac{1}{3}du\right)\right)$$

$$= -\frac{1}{3}\left(\left(-\frac{1}{3}\cos(3x)\right)(\tan(3x)) - \left(-\frac{1}{3}\ln|\tan(u) + \sec(u)|\right)\right) + C$$

$$= -\frac{1}{3}\left(-\frac{1}{3}\cos(3x)\tan(3x) + \frac{1}{3}\ln|\tan(3x) + \sec(3x)|\right)$$

$$u_1 = \frac{1}{9}\cos(3x)\tan(3x) - \frac{1}{9}\ln|\tan(3x) + \sec(3x)|$$

$$u_2 = \int \frac{\cos(3x)\tan(3x)}{3}\,dx \qquad u = 3x \qquad dx = \frac{1}{3}du$$

$$du = 3dx$$

$$= \frac{1}{3}\int \cos(3x)\tan(3x)\,dx$$

$$= \frac{1}{3}\int \cos(u)\tan(u)\frac{1}{3}du$$

$$= \frac{1}{9}\int \cos(u)\tan(u)du$$

$$= \frac{1}{9}\int \sin(u)du$$

$$= -\frac{1}{9}\cos(3x)$$

$$y_p = u_1 y_1 + u_2 y_2$$

$$= \left(\frac{1}{9}\cos(3x)\tan(3x) - \frac{1}{9}\ln|\tan(3x)+\sec(3x)|\right)(\cos(3x)) + \left(\left(-\frac{1}{9}\cos(3x)\right)(\sin(3x))\right)$$

$$= \left(\frac{1}{9}\cos^2(3x)\tan(3x) - \frac{1}{9}\cos(3x)\ln|\tan(3x)+\sec(3x)|\right) - \frac{\sin(6x)}{18}$$

$$= \frac{(2\cos^2(3x)\tan(3x) - 2\cos(3x)\ln|\tan(3x)+\sec(3x)|) - \sin(6x)}{18}$$

$$= \frac{-2\cos(3x)(-\tan(3x)\cos(3x) + \ln|\tan(3x)+\sec(3x)|) - \sin(6x)}{18}$$

<u>Particular Solution:</u>

$$y_p(x) = \frac{-2\cos(3x)(-\tan(3x)\cos(3x) + \ln|\tan(3x + \sec(3x)| - \sin(6x)}{18}$$

General Solution:

$$y = C_1\cos(3x) + C_2\sin(3x) + \frac{-2\cos(3x)\left(-\tan(3x)\cos(3x) + \ln\left|\tan(3x)+\sec(3x)\right|\right) - \sin(6x)}{18}$$