Homework Exercises

to accompany

K. Kuttler – A First Course in Linear Algebra

10th edition

This material includes exercises which first appeared in an earlier edition of *A First Course in Linear Algebra* by K. Kuttler and Lyrix Learning Inc. The current text is available for free at lyryx.com/first-course-linear-algebra/.

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Exercises 1.2 Systems of Equations, Algebraic Procedures

1.2.1 Determine which matrices are in reduced row-echelon form:

$$(a)\begin{bmatrix}1 & 2 & 0\\ 0 & 1 & 7\end{bmatrix} \qquad (b)\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 2\\ 0 & 0 & 0 & 0\end{bmatrix} \qquad (c)\begin{bmatrix}1 & 1 & 0 & 0 & 0 & 5\\ 0 & 0 & 1 & 2 & 0 & 4\\ 0 & 0 & 0 & 0 & 1 & 3\end{bmatrix}$$

1.2.2 Row-reduce the following matrix to obtain the row-echelon form. Then continue to obtain the reduced row-echelon form:

$$\begin{bmatrix}2 & -1 & 3 & -1\\ 1 & 0 & 2 & 1\\ 1 & -1 & 1 & -2\end{bmatrix}$$

1.2.3 Row-reduce the following matrix to obtain the row-echelon form. Then continue to obtain the reduced row-echelon form:

$$\begin{bmatrix}0 & 0 & -1 & -1\\ 1 & 1 & 1 & 0\\ 1 & 1 & 0 & -1\end{bmatrix}$$

1.2.4 Row-reduce the following matrix to obtain the row-echelon form. Then continue to obtain the reduced row-echelon form:

$$\begin{bmatrix}3 & -6 & -7 & -8\\ 1 & -2 & -2 & -2\\ 1 & -2 & -3 & -4\end{bmatrix}$$

1.2.5 Find the solution of the system whose augmented matrix is

$$\left[\begin{array}{ccc|c}1 & 2 & 0 & 2\\ 1 & 3 & 4 & 2\\ 1 & 0 & 2 & 1\end{array}\right]$$

1.2.6 Find the solution of the system of equations:

$9x - 2y + 4z = -17$
$13x - 3y + 6z = -25$
$-2x - z = 3$

1.2.7 Find the solution of the system of equations:

$65x + 84y + 16z = 546$
$81x + 105y + 20z = 682$
$84x + 110y + 21z = 713$

1.2.8 Find the solution of the system of equations:

8x + 2y + 3z = -3
8x + 3y + 3z = -1
4x + y + 3z = -9

1.2.9 Find the solution of the system of equations:

-19x + 8y = -108
-71x + 30y = -404
-2x + y = -12
4x + z = 14

1.2.10 Find the solution of the system whose augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 2 \\ 2 & 0 & 1 & 1 \\ 3 & 2 & 1 & 3 \end{array}\right]$$

1.2.11 Find the solution of the system whose augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 1 & 0 & 4 & 2 \end{array}\right]$$

1.2.12 Find the solution of the system whose augmented matrix is

$$\left[\begin{array}{ccccc|c} 1 & 0 & 2 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 & 2 & 1 \\ 0 & 2 & 0 & 0 & 1 & 3 \\ 1 & -1 & 2 & 2 & 2 & 0 \end{array}\right]$$

1.2.13 Find the solution of the system of equations:

7x + 14y + 15z = 22
2x + 4y + 3z = 5
3x + 6y + 10z = 13

1.2.14 Find the solution of the system of equations:

3x - y + 4z = 6
y + 8z = 0
-2x + y = -4

1.2.15 Find the solution of the system of equations:

$3x - y - 2z = 3$
$y - 4z = 0$
$-2x + y = -2$

Exercises 2.1 Matrix Addition and Scalar Multiplication

2.1.1 For the following pairs of matrices, determine if the sum A + B is defined. If so, find the sum.

(a) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(b) $A = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 & 3 \\ 0 & 1 & 4 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & 0 \\ -2 & 3 \\ 4 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 4 \end{bmatrix}$

2.1.2 For each matrix A, find the product (-2)A, 0A, and 3A.

(a) $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

(b) $A = \begin{bmatrix} -2 & 3 \\ 0 & 2 \end{bmatrix}$

(c) $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & -1 & 3 \\ 4 & 2 & 0 \end{bmatrix}$

Exercises 2.2 Matrix Multiplication

2.2.1 Consider the matrices A = $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 7 \end{bmatrix}$, B = $\begin{bmatrix} 3 & -1 & 2 \\ -3 & 2 & 1 \end{bmatrix}$, C = $\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$, D = $\begin{bmatrix} -1 & 2 \\ 2 & -3 \end{bmatrix}$, E = $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

Find the following if possible. If not possible explain why.

(a) -3A
(b) 3B – A
(c) AC
(d) CB
(e) AE
(f) EA

Exercises 2.3 The Transpose

2.3.1 Let X = $\begin{bmatrix} -1 & -1 & 1 \end{bmatrix}$ and Y = $\begin{bmatrix} 0 & 1 & 2 \end{bmatrix}$. Find X^TY and Y^TX if possible.

2.3.2 Write the system

w – x + y
2y + w
3y
3z + 3x + w

in the form A $\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$, where A is an appropriate matrix.

2.3.3 A matrix A is called **idempotent** if A^2 = A.
Let A = $\begin{bmatrix} 2 & 0 & 2 \\ 1 & 1 & 2 \\ -1 & 0 & -1 \end{bmatrix}$ and show that A is idempotent.

2.3.4 Consider the matrices A = $\begin{bmatrix} 1 & 2 \\ 3 & 2 \\ 1 & -1 \end{bmatrix}$, B = $\begin{bmatrix} 2 & -5 & 2 \\ -3 & 2 & 1 \end{bmatrix}$, C = $\begin{bmatrix} 1 & 2 \\ 5 & 0 \end{bmatrix}$, D = $\begin{bmatrix} -1 & 1 \\ 4 & -3 \end{bmatrix}$, E = $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

Find the following if possible. If it is not possible explain why.

(a) $-3A^T$
(b) $3B - A^T$
(c) E^TB
(d) EE^T
(e) B^TB
(f) CA^T
(g) D^TBE

Exercises 2.5 Finding the Inverse of a Matrix

2.5.1 If $A = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$ find A^{-1} if possible or explain why it does not exist.

2.5.2 If $A = \begin{bmatrix} 0 & 1 \\ 5 & 3 \end{bmatrix}$ find A^{-1} if possible or explain why it does not exist.

2.5.3 If $A = \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix}$ find A^{-1} if possible or explain why it does not exist.

2.5.4 If $A = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$ find A^{-1} if possible or explain why it does not exist.

2.5.5 If $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 1 & 0 & 2 \end{bmatrix}$ find A^{-1} if possible or explain why it does not exist.

2.5.6 If $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 3 & 4 \\ 1 & 0 & 2 \end{bmatrix}$ find A^{-1} if possible or explain why it does not exist.

2.5.7 If $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 4 & 5 & 10 \end{bmatrix}$ find A^{-1} if possible or explain why it does not exist.

2.5.8 If $A = \begin{bmatrix} 1 & 2 & 0 & 2 \\ 1 & 1 & 2 & 0 \\ 2 & 1 & -3 & 2 \\ 1 & 2 & 1 & 2 \end{bmatrix}$ find A^{-1} if possible or explain why it does not exist.

2.5.9 Using the inverse of the matrix, find the solution of the systems:

(a) $\begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

2.5.10 Using the inverse of the matrix, find the solution of the systems:

(a) $\begin{bmatrix} 1 & 0 & 3 \\ 2 & 3 & 4 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 3 \\ 2 & 3 & 4 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$

Exercises 2.6 Elementary Matrices

2.6.1 Let A = $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$. Suppose a row operation is applied to A and the result is B = $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$. Find the elementary matrix E that represents this row operation.

2.6.2 Let A = $\begin{bmatrix} 4 & 0 \\ 2 & 1 \end{bmatrix}$. Suppose a row operation is applied to A and the result is B = $\begin{bmatrix} 8 & 0 \\ 2 & 1 \end{bmatrix}$. Find the elementary matrix E that represents this row operation.

2.6.3 Let A = $\begin{bmatrix} 1 & -3 \\ 0 & 5 \end{bmatrix}$. Suppose a row operation is applied to A and the result is B = $\begin{bmatrix} 1 & -3 \\ 2 & -1 \end{bmatrix}$. Find the elementary matrix E that represents this row operation.

2.6.4 Let A = $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 1 \\ 2 & -1 & 4 \end{bmatrix}$. Suppose a row operation is applied to A and the result is B = $\begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 4 \\ 0 & 5 & 1 \end{bmatrix}$.

(a) Find the elementary matrix E such that EA = B.

(b) Find the inverse of E, E^{-1}, such that $E^{-1}B$ = A.

2.6.5 Let A = $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 1 \\ 2 & -1 & 4 \end{bmatrix}$. Suppose a row operation is applied to A and the result is B = $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 10 & 2 \\ 2 & -1 & 4 \end{bmatrix}$.

(a) Find the elementary matrix E such that EA = B.

(b) Find the inverse of E, E^{-1}, such that $E^{-1}B$ = A.

2.6.6 Let A = $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 1 \\ 2 & -1 & 4 \end{bmatrix}$. Suppose a row operation is applied to A and the result is B = $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 1 \\ 1 & -1/2 & 2 \end{bmatrix}$.

(a) Find the elementary matrix E such that EA = B.

(b) Find the inverse of E, E^{-1}, such that $E^{-1}B$ = A.

2.6.7 Let A = $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 1 \\ 2 & -1 & 4 \end{bmatrix}$. Suppose a row operation is applied to A and the result is B = $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 5 \\ 2 & -1 & 4 \end{bmatrix}$.

(a) Find the elementary matrix E such that EA = B.

(b) Find the inverse of E, E^{-1}, such that $E^{-1}B$ = A.

Exercises 3.1 Basic Techniques and Properties of Determinants

3.1.1 Find the determinants of the following matrices:

(a) $\begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 3 \\ 0 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 4 & 3 \\ 6 & 2 \end{bmatrix}$

3.1.2 Find the determinants of the following matrices

(a) $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 2 \\ 0 & 9 & 8 \end{bmatrix}$

(b) $\begin{bmatrix} 4 & 3 & 2 \\ 1 & 7 & 8 \\ 3 & -9 & 3 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 & 3 & 2 \\ 1 & 3 & 2 & 3 \\ 4 & 1 & 5 & 0 \\ 1 & 2 & 1 & 2 \end{bmatrix}$

3.1.3 Compute the determinant by cofactor expansion. Pick the easiest row or column to use:

$$\begin{vmatrix} 1 & 0 & 0 & 1 \\ 2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 2 & 1 & 3 & 1 \end{vmatrix}$$

3.1.4 Find the determinant using row operations to first simplify:

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ -4 & 1 & 2 \end{vmatrix}$$

3.1.5 Find the determinant using row operations to first simplify:

$$\begin{vmatrix} 2 & 1 & 3 \\ 2 & 4 & 2 \\ 1 & 4 & -5 \end{vmatrix}$$

3.1.6 Find the determinant using row operations to first simplify:

$$\begin{vmatrix} 1 & 2 & 1 & 2 \\ 3 & 1 & -2 & 3 \\ -1 & 0 & 3 & 1 \\ 2 & 3 & 2 & -2 \end{vmatrix}$$

3.1.7 Find the determinant using row operations to first simplify:

$$\begin{vmatrix} 1 & 4 & 1 & 2 \\ 3 & 2 & -2 & 3 \\ -1 & 0 & 3 & 3 \\ 2 & 1 & 2 & -2 \end{vmatrix}$$

Exercises 3.2 Applications of the Determinant

3.2.1 Use Cramer's rule to find the solution to

$x + 2y = 1$
$2x - y = 2$

3.2.2 Use Cramer's rule to find the solution to

$2x + 4y = 1$
$x + y = 2$

3.2.3 Use Cramer's rule to find the solution to

$x + 2y + z = 1$
$2x - y - z = 2$
$x + z = 1$

3.2.4 Use Cramer's rule to find the solution to

$x + 3z = 1$
$2x + 3y + 4z = 0$
$x + 2z = 1$

Exercises 4.2 Vectors in $\mathbf{R}^n$

4.2.1 Find $-3\begin{bmatrix} 5 \\ -1 \\ 2 \\ -3 \end{bmatrix} + 5\begin{bmatrix} -8 \\ 2 \\ -3 \\ 6 \end{bmatrix}$

4.2.2 Find $-7\begin{bmatrix} 6 \\ 0 \\ 4 \\ -1 \end{bmatrix} + 6\begin{bmatrix} -13 \\ -1 \\ 1 \\ 6 \end{bmatrix}$

4.2.3 Decide whether $\mathbf{v} = \begin{bmatrix} 4 \\ 4 \\ -3 \end{bmatrix}$ is a linear combination of the vectors $\mathbf{u}_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$.

4.2.4 Decide whether $\mathbf{v} = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$ is a linear combination of the vectors $\mathbf{u}_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$.

Exercises 4.3 Length of a Vector

4.3.1 Find the distance between the points P and Q where P = (1,2,4) and Q = (3,5,7).

4.3.2 Find the distance between the points P and Q where P = (1,2,4,5) and Q = (-1,3,5,7).

4.3.3 Find the unit vector $\mathbf{u}$ which has the same direction as $\mathbf{v} = \begin{bmatrix} 3 \\ 2 \\ -3 \end{bmatrix}$.

4.3.4 Find the unit vector $\mathbf{u}$ which has the same direction as $\mathbf{v} = \begin{bmatrix} 5 \\ 0 \\ 2 \\ -3 \end{bmatrix}$.

Exercises 4.4 The Dot Product

4.4.1 Find $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \bullet \begin{bmatrix} 2 \\ 0 \\ 1 \\ 3 \end{bmatrix}$.

4.4.2 Find the angle between the vectors $\mathbf{u} = \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$.

4.4.3 Find the angle between the vectors $\mathbf{u} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ -7 \end{bmatrix}$.

4.4.4 Find $\mathrm{proj}_{\mathbf{v}}(\mathbf{w})$ where $\mathbf{w} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

4.4.5 Find $\mathrm{proj}_{\mathbf{v}}(\mathbf{w})$ where $\mathbf{w} = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$.

4.4.6 Find $\mathrm{proj}_{\mathbf{v}}(\mathbf{w})$ where $\mathbf{w} = \begin{bmatrix} 1 \\ 2 \\ -2 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}$.

Exercises 4.5 The Cross Product

4.5.1 Find the area of the triangle determined by the three points (1,2,3), (4,2,0) and (-3,2,1).

4.5.2 Find the area of the triangle determined by the three points (1,0,3), (4,1,0) and (-3,1,1).

4.5.3 Find the area of the triangle determined by the three points (1,2,3), (2,3,4) and (3,4,5). Did something interesting happen here? What does it mean geometrically?

4.5.4 Find the area of the parallelogram determined by the vectors

$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$.

4.5.5 Find the area of the parallelogram determined by the vectors

$\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}$.

Exercises 4.6 Parametric Lines

4.6.1 Find the vector equation for the line through (-7,6,0) and (-1,1,4). Then find the parametric equations for this line.

4.6.2 Find parametric equations for the line through the point (7,7,1) with a direction vector $\mathbf{d} = \begin{bmatrix} 1 \\ 6 \\ 2 \end{bmatrix}$.

4.6.3 If the parametric equations for a line are $x = t + 2$, $y = 6 - 3t$ and $z = -t - 6$, find a direction vector for the line and a point on the line.

4.6.4 Find the vector equation for the line through the two points (-5,5,1) and (2,2,4). Then find the parametric equations.

4.6.5 The equation of a line in two dimensions is written as $y = x - 5$. Find parametric equations for this line.

4.6.6 Find parametric equations for the line through (6,5,-2) and (5,1,2).

4.6.7 Find the vector equation and parametric equations for the line through the point (-7,10,-6) with a direction vector $\mathbf{d} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$.

4.6.8 If the parametric equations for a line are $x = 2t + 2$, $y = 5 - 4t$ and $z = -t - 3$, find a direction vector for the line and a point on the line and write the vector equation of the line.

4.6.9 Find the vector equation and parametric equations for the line through the two points (4,10,0) and (1,-5,-6).

Exercises 4.7 Planes in $\mathbf{R}^3$

4.7.1 Find the equation of the plane passing through the point (2,1,2) with normal $\mathbf{n}$ = (1,0,0).

4.7.2 Find the equation of the plane passing through the point (1,0,-3) with normal $\mathbf{n}$ = (0,0,1).

4.7.3 Find the equation of the plane passing through the point (3,2,2) with normal $\mathbf{n}$ = (2,3,-1).

4.7.4 Find the equation of the plane passing through the point (0,0,0) with normal $\mathbf{n}$ = (-3,0,2).

Exercises 4.8 Spanning and Linear Independence in $\mathbf{R}^n$

4.8.1 Determine if $\mathbf{v} = \begin{bmatrix} 2 \\ -3 \\ -4 \end{bmatrix}$ is in the span of the vectors $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix}$, and $\mathbf{u}_4 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$. If it is, then write $\mathbf{v}$ as a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ and $\mathbf{u}_4$, using as few vectors as possible in the linear combination.

4.8.2 Determine if $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ is in the span of the vectors $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix}$, and $\mathbf{u}_4 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$. If it is, then write $\mathbf{v}$ as a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ and $\mathbf{u}_4$, using as few vectors as possible in the linear combination.

4.8.3 Determine if $\mathbf{v} = \begin{bmatrix} 1 \\ 9 \\ 1 \end{bmatrix}$ is in the span of the vectors $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix}$, and $\mathbf{u}_4 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$. If it is, then write $\mathbf{v}$ as a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ and $\mathbf{u}_4$, using as few vectors as possible in the linear combination.

4.8.4 Determine if the vectors $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 3 \\ -1 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 4 \\ -1 \\ 1 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ 4 \\ 0 \\ 1 \end{bmatrix}$, and $\mathbf{u}_4 = \begin{bmatrix} 1 \\ 10 \\ 2 \\ 1 \end{bmatrix}$ are linearly independent. If they are, explain why and if they are not, exhibit one of them as a linear combination of the others. Also give a linear independent set of vectors which have the same span as the given vectors.

4.8.5 Determine if the vectors $\mathbf{u}_1 = \begin{bmatrix} -1 \\ -2 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -3 \\ -4 \\ 3 \\ 3 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 0 \\ -1 \\ 4 \\ 3 \end{bmatrix}$, and $\mathbf{u}_4 = \begin{bmatrix} 0 \\ -1 \\ 2 \\ 1 \end{bmatrix}$ are linearly independent. If they are, explain why and if they are not, exhibit one of them as a linear combination of the others. Also give a linear independent set of vectors which have the same span as the given vectors.

4.8.6 Determine if the vectors $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 5 \\ -2 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 6 \\ -3 \\ 1 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} -1 \\ -4 \\ 1 \\ -1 \end{bmatrix}$, and $\mathbf{u}_4 = \begin{bmatrix} 1 \\ 6 \\ -2 \\ 1 \end{bmatrix}$ are linearly independent. If they are, explain why and if they are not, exhibit one of them as a linear combination of the others. Also give a linear independent set of vectors which have the same span as the given vectors.

Exercises 4.9 Subspaces, Bases, and Dimension

4.9.1 Find a basis for the span of the following 5 vectors in $\mathbf{R}^4$, that is find a linearly independent subset of these vectors which has the same span as these vectors:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix},\ \mathbf{v}_2 = \begin{bmatrix} 1 \\ 3 \\ -3 \\ 1 \end{bmatrix},\ \mathbf{v}_3 = \begin{bmatrix} 1 \\ 3 \\ -2 \\ 1 \end{bmatrix},\ \mathbf{v}_4 = \begin{bmatrix} 4 \\ 3 \\ -1 \\ 4 \end{bmatrix},\ \mathbf{v}_5 = \begin{bmatrix} 1 \\ 3 \\ -2 \\ 1 \end{bmatrix}$$

4.9.2 Find a basis for the span of the following 5 vectors in $\mathbf{R}^4$, that is find a linearly independent subset of these vectors which has the same span as these vectors:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix},\ \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix},\ \mathbf{v}_3 = \begin{bmatrix} 1 \\ -2 \\ -3 \\ 1 \end{bmatrix},\ \mathbf{v}_4 = \begin{bmatrix} 2 \\ -5 \\ -7 \\ 2 \end{bmatrix},\ \mathbf{v}_5 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}$$

4.9.3 Find a basis for the span of the following 5 vectors in $\mathbf{R}^4$, that is find a linearly independent subset of these vectors which has the same span as these vectors:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -2 \\ 1 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} 1 \\ 3 \\ -3 \\ 1 \end{bmatrix}, \ \mathbf{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \ \mathbf{v}_4 = \begin{bmatrix} 2 \\ -3 \\ 3 \\ 2 \end{bmatrix}, \ \mathbf{v}_5 = \begin{bmatrix} 1 \\ 3 \\ -2 \\ 1 \end{bmatrix}$$

4.9.4 Find a basis for the span of the following 5 vectors in $\mathbf{R}^4$, that is find a linearly independent subset of these vectors which has the same span as these vectors:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 4 \\ -2 \\ 1 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} 1 \\ 5 \\ -3 \\ 1 \end{bmatrix}, \ \mathbf{v}_3 = \begin{bmatrix} 1 \\ 5 \\ -2 \\ 1 \end{bmatrix}, \ \mathbf{v}_4 = \begin{bmatrix} 4 \\ 11 \\ -1 \\ 4 \end{bmatrix}, \ \mathbf{v}_5 = \begin{bmatrix} 1 \\ 5 \\ -2 \\ 1 \end{bmatrix}$$

4.9.5 Let $H = \operatorname{span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -2 \\ -2 \end{bmatrix} \right\}$. Find the dimension of H and determine a basis.

4.9.6 Let $H = \operatorname{span}\left\{ \begin{bmatrix} -1 \\ 1 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} -4 \\ 3 \\ -2 \\ -4 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -2 \\ -4 \end{bmatrix}, \begin{bmatrix} -7 \\ 5 \\ -3 \\ -6 \end{bmatrix} \right\}$. Find the dimension of H and determine a basis.

Exercises 4.10 Row Space, Column Space and Null Space of a Matrix

4.10.1 Find the rank of the following matrix. Also, find a basis for the row and column spaces.

$$\begin{bmatrix} 1 & 3 & 0 & -2 & 0 & 3 \\ 3 & 9 & 1 & -7 & 0 & 8 \\ 1 & 3 & 1 & -3 & 1 & -1 \\ 1 & 3 & -1 & -1 & -2 & 10 \end{bmatrix}$$

4.10.2 Find the rank of the following matrix. Also, find a basis for the row and column spaces.

$$\begin{bmatrix} 1 & 3 & 0 & -2 & 7 & 3 \\ 3 & 9 & 1 & -7 & 23 & 8 \\ 1 & 3 & 1 & -3 & 9 & 2 \\ 1 & 3 & -1 & -1 & 5 & 4 \end{bmatrix}$$

4.10.3 Find the rank of the following matrix. Also, find a basis for the row and column spaces.

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 7 & 0 \\ 3 & 1 & 10 & 0 & 23 & 0 \\ 1 & 1 & 4 & 1 & 7 & 0 \\ 1 & -1 & 2 & -2 & 9 & 1 \end{bmatrix}$$

4.10.4 Find the rank of the following matrix. Also, find a basis for the row and column spaces.

$$\begin{bmatrix} 1 & 0 & 3 \\ 3 & 1 & 10 \\ 1 & 1 & 4 \\ 1 & -1 & 2 \end{bmatrix}$$

4.10.5 Find the rank of the following matrix. Also, find a basis for the row and column spaces.

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 3 & 1 & 10 & 0 \\ -1 & 1 & -2 & 1 \\ 1 & -1 & 2 & -2 \end{bmatrix}$$

4.10.6 Find ker (A) for the following matrices:

(a) $A = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix}$

(c) $A = \begin{bmatrix} 2 & 4 & 0 \\ 3 & 6 & -2 \\ 1 & 2 & -2 \end{bmatrix}$ (d) $A = \begin{bmatrix} 2 & -1 & 3 & 5 \\ 2 & 0 & 1 & 2 \\ 6 & 4 & -5 & -6 \\ 0 & 2 & -4 & -6 \end{bmatrix}$

Exercises 4.11 Orthogonal and Orthonormal Sets and Matrices

4.11.1 Determine whether the following set of vectors is orthogonal. If it is orthogonal, determine whether it is also orthonormal.

$$\begin{bmatrix} \frac{1}{6}\sqrt{2}\sqrt{3} \\ \frac{1}{3}\sqrt{2}\sqrt{3} \\ -\frac{1}{6}\sqrt{2}\sqrt{3} \end{bmatrix}, \begin{bmatrix} \frac{1}{2}\sqrt{2} \\ 0 \\ \frac{1}{2}\sqrt{2} \end{bmatrix}, \begin{bmatrix} -\frac{1}{3}\sqrt{3} \\ \frac{1}{3}\sqrt{3} \\ \frac{1}{3}\sqrt{3} \end{bmatrix}$$

If the set of vectors is orthogonal but not orthonormal, give an orthonormal set of vectors which has the same span.

4.11.2 Determine whether the following set of vectors is orthogonal. If it is orthogonal, determine whether it is also orthonormal.

$$\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

If the set of vectors is orthogonal but not orthonormal, give an orthonormal set of vectors which has the same span.

4.11.3 Determine whether the following set of vectors is orthogonal. If it is orthogonal, determine whether it is also orthonormal.

$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

If the set of vectors is orthogonal but not orthonormal, give an orthonormal set of vectors which has the same span.

4.11.4 Determine whether the following set of vectors is orthogonal. If it is orthogonal, determine whether it is also orthonormal.

$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

If the set of vectors is orthogonal but not orthonormal, give an orthonormal set of vectors which has the same span.

4.11.5 Determine whether the following set of vectors is orthogonal. If it is orthogonal, determine whether it is also orthonormal.

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

If the set of vectors is orthogonal but not orthonormal, give an orthonormal set of vectors which has the same span.

4.11.6 Determine if any of the following matrices are symmetric, skew symmetric or orthogonal:

(a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 2 & -3 \\ 2 & 1 & 4 \\ -3 & 4 & 7 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & -2 & -3 \\ 2 & 0 & -4 \\ 3 & 4 & 0 \end{bmatrix}$

4.11.7 Determine whether the matrix is orthogonal, and if so, find its inverse.

(a) $\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ $\qquad$ (b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$

Exercises 5.1 Linear Transformations

5.1.1 Show that each function T is a linear transformation from $\mathbf{R}^3$ to $\mathbf{R}^2$ and determine for each the matrix A such that

T($\begin{bmatrix} x \\ y \\ z \end{bmatrix}$) = A $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

(a) T($\begin{bmatrix} x \\ y \\ z \end{bmatrix}$) = $\begin{bmatrix} x + 2y + 3z \\ 2x - 3y + z \end{bmatrix}$

(b) T($\begin{bmatrix} x \\ y \\ z \end{bmatrix}$) = $\begin{bmatrix} 7x + 2y + z \\ 3x - 11y + 2z \end{bmatrix}$

(c) T($\begin{bmatrix} x \\ y \\ z \end{bmatrix}$) = $\begin{bmatrix} 3x + 2y + z \\ x + 2y + 6z \end{bmatrix}$

(d) T($\begin{bmatrix} x \\ y \\ z \end{bmatrix}$) = $\begin{bmatrix} -5x + 2y + z \\ x + y + z \end{bmatrix}$

Exercises 7.1 Eigenvalues and Eigenvectors of a Matrix

7.1.1 Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} -6 & -92 & 12 \\ 0 & 0 & 0 \\ -2 & -31 & 4 \end{bmatrix}$$

7.1.2 Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} -2 & -17 & -6 \\ 0 & 0 & 0 \\ 1 & 9 & 3 \end{bmatrix}$$

7.1.3 Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 9 & 2 & 8 \\ 2 & -6 & -2 \\ -8 & 2 & -5 \end{bmatrix}$$

7.1.4 Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 6 & 76 & 16 \\ -2 & -21 & -4 \\ 2 & 64 & 17 \end{bmatrix}$$

Exercises 7.2 Diagonalization

7.2.1 Find the eigenvalues and eigenvectors of the matrix. One eigenvalue is 1. Diagonalize if possible.

$$\begin{bmatrix} 5 & -18 & -32 \\ 0 & 5 & 4 \\ 2 & -5 & -11 \end{bmatrix}$$

7.2.2 Find the eigenvalues and eigenvectors of the matrix. One eigenvalue is 3. Diagonalize if possible.

$$\begin{bmatrix} -13 & -28 & 28 \\ 4 & 9 & -8 \\ -4 & -8 & 9 \end{bmatrix}$$

7.2.3 Find the eigenvalues and eigenvectors of the matrix. One eigenvalue is -3. Diagonalize if possible.

$$\begin{bmatrix} 89 & 38 & 268 \\ 14 & 2 & 40 \\ -30 & -12 & -90 \end{bmatrix}$$

7.2.4 Find the eigenvalues and eigenvectors of the matrix. One eigenvalue is 1. Diagonalize if possible.

$$\begin{bmatrix} 1 & 90 & 0 \\ 0 & -2 & 0 \\ 3 & 89 & -2 \end{bmatrix}$$

Exercises 7.3 Raising a Matrix to a Higher Power

7.3.1 Let A = $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. Diagonalize A to find A^{10}.

7.3.2 Let A = $\begin{bmatrix} 1 & 4 & 1 \\ 0 & 2 & 5 \\ 0 & 0 & 5 \end{bmatrix}$. Diagonalize A to find A^5.

7.3.3 Let A = $\begin{bmatrix} 1 & -2 & -1 \\ 2 & -1 & 1 \\ -2 & 3 & 1 \end{bmatrix}$. Diagonalize A to find A^5.

Exercises 7.4 Orthogonal Diagonalization and Quadratic Forms

7.4.1 Find the eigenvalues and an orthonormal basis of eigenvectors for A. Hint: One eigenvalue is 3.

$$A = \begin{bmatrix} 4 & 1 & -2 \\ 1 & 4 & -2 \\ -2 & -2 & 7 \end{bmatrix}$$

7.4.2 Find the eigenvalues and an orthonormal basis of eigenvectors for A. Diagonalize A by finding an orthogonal matrix U and a diagonal matrix D such that $U^TAU = D$. Hint: One eigenvalue is -2.

$$A = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

7.4.3 Find the eigenvalues and an orthonormal basis of eigenvectors for A. Diagonalize A by finding an orthogonal matrix U and a diagonal matrix D such that $U^TAU = D$.

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3/2 & 1/2 \\ 0 & 1/2 & 3/2 \end{bmatrix}$$

7.4.4 Find the eigenvalues and an orthonormal basis of eigenvectors for A. Diagonalize A by finding an orthogonal matrix U and a diagonal matrix D such that $U^TAU = D$.

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 1 \\ 0 & 1 & 5 \end{bmatrix}$$

7.4.5 Consider the quadratic form given by $q = -2x_1^2 + 2x_1x_2 - 2x_2^2$.

(a) Write q in the form $\mathbf{x}^T A\mathbf{x}$ for an appropriate symmetric matrix A and vector $\mathbf{x}$.

(b) Use a change of variables to rewrite q to eliminate the x_1x_2 term.

(c) Find the angle of rotation of the old axes to the new axes.

7.4.6 Consider the quadratic form given by $q = 7x_1^2 + 6x_1x_2 - x_2^2$.

(a) Write q in the form $\mathbf{x}^T A\mathbf{x}$ for an appropriate symmetric matrix A and vector $\mathbf{x}$.

(b) Use a change of variables to rewrite q to eliminate the x_1x_2 term.

(c) Find the angle of rotation of the old axes to the new axes.