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# Taylor Polynomials: Problem 1

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Find the fourth-degree Taylor polynomial $T_4(x)$ for the function $f(x) = \cos x$ at the number $x = \frac{\pi}{4}$.

Answer: $T_4(x) =$ ☐

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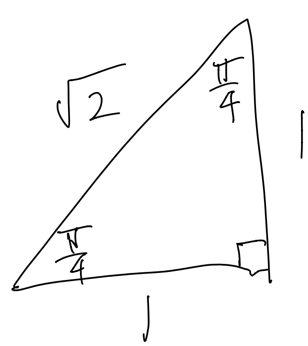
$$T_4(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4$$




$$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$f(x) = \cos(x)$ $\qquad$ $f\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

$f'(x) = -\sin(x)$ $\qquad$ $f'\left(\frac{\pi}{4}\right) = -\sin\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$

$f''(x) = -\cos(x)$ $\qquad$ $f''\left(\frac{\pi}{4}\right) = -\cos\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$

$f'''(x) = \sin(x)$ $\qquad$ $f'''\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

$f^{(4)}(x) = \cos(x)$ $\qquad$ $f^{(4)}\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

$$T_4(x) = \frac{1}{\sqrt{2}} + -\frac{1}{\sqrt{2}}\left(x-\frac{\pi}{4}\right) + \frac{-\frac{1}{\sqrt{2}}}{2!}\left(x-\frac{\pi}{4}\right)^2 + \frac{\frac{1}{\sqrt{2}}}{3!}\left(x-\frac{\pi}{4}\right)^3 + \frac{\frac{1}{\sqrt{2}}}{4!}\left(x-\frac{\pi}{4}\right)^4$$

$$= \underbrace{\frac{1}{\sqrt{2}}}_{A} \underbrace{-\frac{1}{\sqrt{2}}}_{B}\left(x-\frac{\pi}{4}\right) \underbrace{-\frac{1}{2\sqrt{2}}}_{C}\left(x-\frac{\pi}{4}\right)^2 + \underbrace{\frac{1}{6\sqrt{2}}}_{D}\left(x-\frac{\pi}{4}\right)^3 + \underbrace{\frac{1}{24\sqrt{2}}}_{E}\left(x-\frac{\pi}{4}\right)^4$$

---

$$\frac{-\frac{1}{\sqrt{2}}}{2} = \frac{-\frac{1}{\sqrt{2}}}{\frac{2}{1}} = -\frac{1}{\sqrt{2}} \cdot \frac{1}{2} = -\frac{1}{2\sqrt{2}}$$

# 5.1 SEQUENCES

ONE MOTIVATION:

Taylor polynomials - the higher the degree, the better the approximation

"BEST APPROXIMATION"

Would be an "infinite degree polynomial"

These don't exist BUT instead we use "power series"

$$T(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots$$

FIRST: Rather than adding up infinitely many powers of $x$, let's start by adding up infinitely many <u>numbers</u>.

This is called a "series"

FIRST: Discuss <u>lists</u> of infinitely many numbers (not adding them up)
This is called a "sequence"

EXAMPLES

(a) $\{1, 8, 6, -3, 0, \pi, 0.1, \ldots\ldots\} = a_n$

$a_1 = 1, \; a_2 = 8, \; a_3 = 6, \ldots$ probably diverges

(b) $a_1 = 1, \; a_2 = 1, \; a_n = a_{n-1} + a_{n-2}$

Fibonacci sequence:

$1, 1, 2, 3, 5, 8, 13, 21, \ldots\ldots$

$n=3$: $a_3 = a_{3-1} + a_{3-2}$
$= a_2 + a_1$
$= 1+1$
$= 2$

$n=4$: $a_4 = a_{4-1} + a_{4-2}$
$= a_3 + a_2$
$= 2+1$
$= 3$

diverges

(c) $a_1 = 0 \qquad a_{n+1} = 2a_n + 5 \;//\; a_n = 2a_{n-1} + 5$

$0, 5, 15, 35, \ldots\ldots$ diverges

$a_2 = 2a_1 + 5 \qquad a_3 = 2a_2 + 5$
$= 2 \cdot 0 + 5 \qquad = 2 \cdot 5 + 5$
$= 5 \qquad = 15$

$a_4 = 2a_3 + 5$
$= 2 \cdot 15 + 5$ "recursively defined"
$= 35$

(d) $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}, \frac{1}{256} \cdots$

$a_n = \left(\frac{1}{2}\right)^n = \frac{1}{2^n}$ "closed form"

$a_{200} = \left(\frac{1}{2}\right)^{200} = \ldots$

this sequence "converges" to 0

WE WILL BE INTERESTED IN WHETHER A SEQUENCE "CONVERGES"

Def: The sequence $a_n$ converges to $L$ if

$$\lim_{n \to \infty} a_n = L$$

In example (d)

$$\lim_{n \to \infty} \frac{1}{2^n} = 0$$

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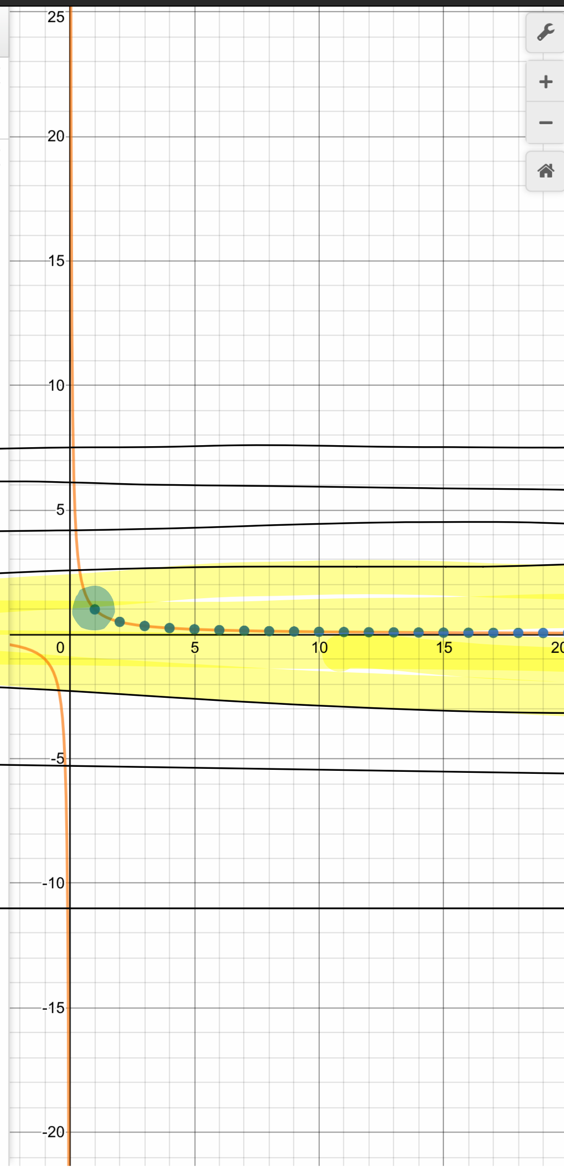
f(x)=1/x function and a_n=… Save desmos Kate

$$f(x) = \frac{1}{x}$$

| $x_1$ | $f(x_1)$ |
|---|---|
| 1 | 1 |
| 2 | 0.5 |
| 3 | 0.33333333 |
| 4 | 0.25 |
| 5 | 0.2 |
| 6 | 0.16666667 |
| 7 | 0.14285714 |
| 8 | 0.125 |
| 9 | 0.11111111 |
| 10 | 0.1 |
| 11 | 0.090909091 |
| 12 | 0.083333333 |
| 13 | 0.076923077 |
| 14 | 0.071428571 |
| 15 | 0.066666667 |
| 16 | 0.0625 |
| 17 | 0.058823529 |
| 18 | 0.055555556 |
| 19 | 0.052631579 |
| 20 | 0.05 |
| 21 | 0.047619048 |
| 22 | 0.045454545 |
| 23 | 0.043478261 |
| 24 | 0.041666667 |
| 25 | 0.04 |




EXAMPLE (e)

$$a_n = \frac{1}{n}$$

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8} \ldots$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n}$$

$$= \lim_{x\to\infty} \frac{1}{x}$$

$$= 0$$

$f(x) = \frac{1}{x}$ has a horizontal asymptote as $x \to \infty$

equation $y = 0$

so the sequence $a_n = \frac{1}{n}$ converges to 0

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EXAMPLE (f)

$a_n = \cos(2n\pi) = \cos(2\pi n)$ (sequence)

corresponding function

$f(x) = \cos(2\pi x)$

no horizontal asymptote

$a_n = \cos(2n\pi)$

$a_1 = \cos(2\pi)$ $\quad a_2 = \cos(2\cdot 2\pi)$ $\quad a_3 = \cos(2\cdot 3\pi)$

$= 1$ $\quad = \cos(4\pi)$ $\quad = \cos(6\pi)$

$= 1$ $\quad = 1$

$1, 1, 1, 1, 1, \ldots\ldots$

really: compare this sequence to the function $f(x) = 1$

$\lim_{x \to \infty} 1 = 1$

does have a H.A. $y = 1$

so sequence

$a_n = \cos(2n\pi)$

$1, 1, 1, 1, \ldots\ldots$

converges to 1

FACT: Bounded monotonic sequences converge

either increasing or decreasing

EXAMPLE (e) $a_n = \frac{1}{n}$

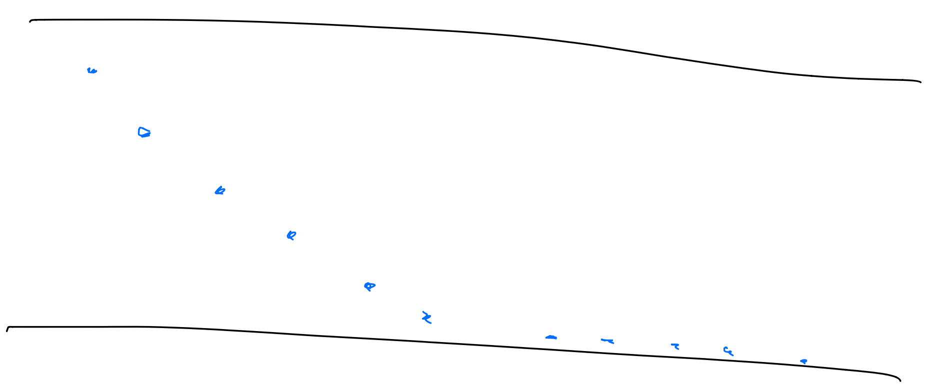
"BOUNDED BELOW"

means there is a horizontal line the sequence stays above

"BOUNDED ABOVE"

means there is a horizontal line the sequence stays below

$a_n = \frac{1}{n}$ is monotonic - decreasing



If a sequence does not converge, we say it diverges

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OFFICE HOURS

$$\int \frac{1}{x^2+\cancel{5}\,9}\,dx = \int \frac{1}{x^2+(\sqrt{5})^2}\,dx$$

$$= \int \frac{1}{5\sec^2\theta}\sqrt{5}\sec^2\theta\,d\theta$$

$x = a\tan(\theta)$

$a=\sqrt{5}$

$x=\sqrt{5}\tan(\theta) \longrightarrow dx=\sqrt{5}\sec^2\theta\,d\theta$

$x^2 = 5\tan^2\theta$

$x^2+5 = 5\tan^2\theta+5$

$= 5(\tan^2\theta+1)$

$= 5\sec^2\theta$

$$= \int \frac{\sqrt{5}}{5}\frac{\sec^2\theta}{\sec^2\theta}\,d\theta$$

$$\frac{\sin^2\theta + \cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$\tan^2\theta + 1 = \sec^2\theta$$

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$$\int 1\,dx = x + C$$

$$\frac{d}{d\theta}(\theta) = 1$$

$$\frac{d}{dx}(x) = 1$$

$$= \frac{\sqrt{5}}{5}\int 1\,d\theta$$

$$= \frac{\sqrt{5}}{5}\theta + C$$

$$= \frac{\sqrt{5}}{5}\tan^{-1}\left(\frac{x}{\sqrt{5}}\right) + C$$

$$x = \sqrt{5}\tan\theta$$

$$\frac{x}{\sqrt{5}} = \tan\theta$$

$$\tan^{-1}\left(\frac{x}{\sqrt{5}}\right) = \theta$$

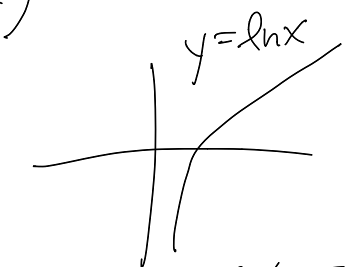
---

$$\int_0^\infty \frac{1}{x+5}\,dx = \lim_{b\to\infty}\int_0^b \frac{1}{x+5}\,dx$$

$$= \lim_{b\to\infty} \ln(|x+5|)\Big|_0^b$$

$$= \lim_{b\to\infty} \ln(|b+5|) - \ln(|0+5|)$$

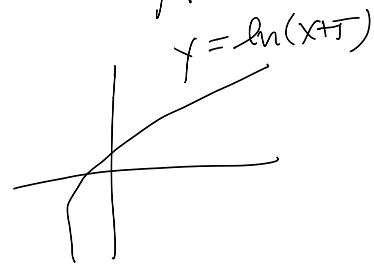
$$= \lim_{b\to\infty} \ln(b+5) - \ln(5)$$




does not exist

So

$$\int_0^\infty \frac{1}{x+5}\,dx \text{ diverges}$$

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$$\int_0^\infty \frac{1}{x^2+5}\,dx = \lim_{b\to\infty} \int_0^b \frac{1}{x^2+5}\,dx$$

$$= \lim_{b\to\infty} \frac{\sqrt{5}}{5} \tan^{-1}\left(\frac{x}{\sqrt{5}}\right)\Bigg|_0^b$$

$\frac{\pi}{2}$

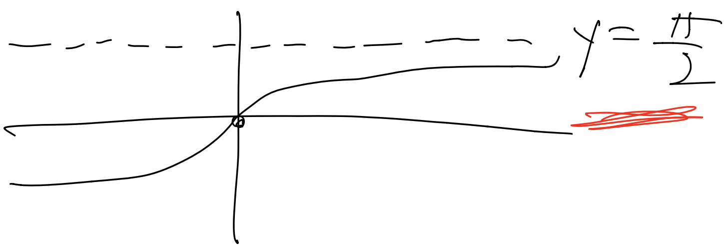
$\infty$

$$= \lim_{b\to\infty} \frac{\sqrt{5}}{5}\tan^{-1}\left(\frac{b}{\sqrt{5}}\right) - \frac{\sqrt{5}}{5}\tan^{-1}(0)$$

$$= \frac{\sqrt{5}}{5}\cdot\frac{\pi}{2} - 0$$

$$= \frac{\sqrt{5}\pi}{10}$$

$y = \tan^{-1}\left(\frac{x}{\sqrt{5}}\right)$




as $x\to\infty$

$\frac{x}{\sqrt{5}} \to \infty$

So $\displaystyle\int_0^\infty \frac{1}{x^2+5}\,dx$ converges to

$$\frac{\sqrt{5}\pi}{10}$$

$$\int \frac{-4}{(x-1)(x^2-9)}\,dx \qquad (x^2-9) = (x+3)(x-3)$$

$$\cancel{\frac{A}{x-1} + \frac{Bx+C}{x^2-9} = Ax^2 - A9 + (x-1)\cdots}$$

$$= -4\int \frac{1}{(x-1)(x+3)(x-3)}\,dx$$

---

$$\int \frac{1}{(x-1)(x+3)^2}\,dx$$

$$\frac{1}{(x-1)(x+3)^2} = \frac{A}{x-1} + \frac{B}{x+3} + \frac{C}{(x+3)^2}$$

$$\frac{1}{(x-1)(x+3)^2} = \frac{A(x+3)^2}{(x-1)(x+3)^2} + \frac{B(x-1)(x+3)}{(x-1)(x+3)^2} + \frac{C(x-1)}{(x-1)(x+3)^2}$$

$$\frac{1}{(x-1)(x+3)^2} = \frac{A(x+3)^2 + B(x-1)(x+3) + C(x-1)}{(x-1)(x+3)^2}$$

$$1 = A(x+3)^2 + B(x-1)(x+3) + C(x-1)$$

$$1 = A(x^2+6x+9) + B(x^2+2x-3) + C(x-1)$$

$$1 = Ax^2 + 6Ax + 9A + Bx^2 + 2Bx - 3B + Cx - C$$

$$0x^2 + 0x + 1 = (A+B)x^2 + (6A+2B+C)x + 9A - 3B - C$$

$$\left.\begin{array}{ll} ① & A+B = 0 \\ ② & 6A+2B+C = 0 \\ ③ & 9A-3B-C = 1 \end{array}\right\}$$ solve for $A, B, C$

$$
\begin{array}{lr}
② & 15A - B = 1 \\
+③ & \\
+① & A + B = 0 \\
\hline
 & 16A = 1
\end{array}
$$

$$A = \frac{1}{16}$$

①

$$\frac{1}{16} + B = 0$$

$$B = -\frac{1}{16}$$

② $6\left(\frac{1}{16}\right) + 2\left(-\frac{1}{16}\right) + C = 0$

$$\frac{6}{16} - \frac{2}{16} + C = 0$$

$$\frac{4}{16} + C = 0$$

$$\frac{1}{4} + C = 0$$

$$C = -\frac{1}{4}$$