MAT 1475
Fall 2014
Professor K. Poirier
Test #3
December 1, 2014

Name (Print): SOLUTIONS

Time Limit: 100 Minutes

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This exam contains 5 pages and 5 problems. Check to see if any pages are missing. Print your name on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books or notes on this exam. You may use a calculator.

**Total: 50 points**, including 2 points overall for clear, precise, effective, and complete communication and justification (this includes correct use of all symbols and notation)

**STYLE:**

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1. (5 points) Determine the equation of the tangent line to the graph of

$$y^2 - 7xy + x^3 - 2x = 9$$

at the point $(0, 3)$. Leave your answer in slope-$y$-intercept form.

Implicit differentiation

$$2y\frac{dy}{dx} - \left(7y + 7x\frac{dy}{dx}\right) + 3x^2 - 2 = 0$$

at $(x,y) = (0,3)$

$$2\cdot 3\frac{dy}{dx} - \left(7\cdot 3 + 7\cdot \cancel{0\cdot\frac{dy}{dx}}\right) + 3\cdot\cancel{0^2} - 2 = 0$$

$$6\frac{dy}{dx} = 2 + 21$$

$$\frac{dy}{dx} = \frac{23}{6}$$

equation of line:

$$y - y_1 = m(x - x_1)$$

$y_1 = 3$, $m = \frac{23}{6}$, $x_1 = 0$

$$y = \frac{23}{6}x + 3$$

2. (5 points) Determine the maximum and minimum values of the function $f(x) = x^3 - 3x^2 - 1$ on the interval $[1, 4]$.

$f'(x) = 3x^2 - 6x$

no type (2) critical points

type (1) solve $3x^2 - 6x = 0$

$3x(x-2) = 0$

$x = 0$ or $x = 2$

exclude since not in $[1,4]$

$f(1) = 1^3 - 3\cdot 1^2 - 1$
$= -3$

$f(2) = 2^3 - 3\cdot 2^2 - 1$
$= -5$ ← minimum value

$f(4) = 4^3 - 3\cdot 4^2 - 1$
$= 15$ ← maximum value

3. (5 points) Use linear approximation to approximate $\sqrt{16.4}$. Do not use a calculator.

Let $f(x) = \sqrt{x}$ $a = 16$ $f(16) = 4$

$f'(x) = \frac{1}{2\sqrt{x}}$

$f'(16) = \frac{1}{8}$

so $L(x) = \frac{1}{8}(x-16) + 4$

and $L(16.4) = \frac{1}{8}(16.4 - 16) + 4$
$= 4.05$

so $\sqrt{16.4} \approx 4.05$

4. (5 points) Sand pours from a chute and forms a conical pile whose height is always equal to its base diameter. The height of the pile increases at a rate of 5 feet per hour. Find the rate at which the volume of sand in the conical pile is changing when the height of the pile is 4 feet. (Hint, the volume of a cone is given by $V = \frac{1}{3}\pi r^2 h$.)

$h = 2r$

$h = 4 \Rightarrow r = 2$

$V = \frac{1}{3}\pi r^2 h$

$V = \frac{1}{3}\pi r^2 \cdot 2r$

$V = \frac{2}{3}\pi r^3$

$\frac{dh}{dt} = 2\frac{dr}{dt}$

$5 = 2 \cdot 2.5$

$\frac{dV}{dt} = \frac{2}{\cancel{3}}\pi \cdot \cancel{3} r^2 \frac{dr}{dt}$

$\left.\frac{dV}{dt}\right|_{r=2} = 2\pi \cdot 2^2 (2.5) =$ ~~[illegible]~~ $20\pi$ $ft^3/s$

5. Let

$$f(x) = \frac{x^2}{(x-2)^2}.$$

(a) (1 point) Determine the domain of $f(x)$.

$(x-2)^2 \neq 0$

so $D = \{x \in \mathbb{R} \mid x \neq 2\}$

(b) (1 point) Determine any $x$-values where $f(x)$ is discontinuous.

f(x) is a rational function, which means it's continuous on its domain, so the only point where f(x) is discontinuous is at $x=2$

(c) (2 points) Determine the $x$ and $y$ intercepts of the graph of $f(x)$.

<u>y-intercept</u>

$y = f(0) = \frac{0^2}{(0-2)^2} = 0$

<u>x-intercept</u>

solve $f(x)=0$

$\frac{x^2}{(x-2)^2} = 0$

$x^2 = 0$

$x = 0$

(d) (3 points) Use limits to determine the equations of all vertical asymptotes, if they exist, and the behavior of the function near those asymptotes.

Check limits on both sides of the point of discontinuity:

$\lim_{x \to 2^-} \frac{x^2}{(x-2)^2} = \infty$ $\quad$ $\lim_{x \to 2^+} \frac{x^2}{(x-2)^2} = \infty$

so $x=2$ is the equation of the V.A.

(e) (3 points) Use limits to determine the equations of all horizontal asymptotes, if they exist.

$\lim_{x \to \infty} \frac{x^2}{(x-2)^2} = \lim_{x \to \infty} \frac{x^2}{x^2-4x+4} = \lim_{x \to \infty} \frac{\frac{x^2}{x^2}}{\frac{x^2}{x^2} - \frac{4x}{x^2} + \frac{4}{x^2}} = \frac{1}{1-0+0} = 1$

Similarly $\lim_{x \to -\infty} \frac{x^2}{(x-2)^2} = 1$

so $y=1$ is the equation of the H.A.

(f) (4 points) Determine the intervals where $f(x)$ is increasing or decreasing.

$$f(x)=\frac{x^2}{(x-2)^2}$$

$$f'(x)=\frac{2x(x-2)^2-x^2\cdot 2(x-2)}{(x-2)^4}=\frac{(x-2)\left[2x^2-4x-2x^2\right]}{(x-2)^4}$$

$$=\cancel{2x(x^2-4x+4)-2x^2(x-2)}=\frac{-4x}{(x-2)^3}$$

$(-\infty,0)$ 0 $(0,2)$ 2 $(2,\infty)$

sign of $f'(x)$: $-$ $+$ $-$

behavior of $f(x)$: ↘ ↗ ↘

increasing $(0,2)$

decreasing $(-\infty,0)\cup(2,\infty)$

(g) (2 points) Determine the $x$ and $y$ coordinates of any local maxima or minima.

min $(0,0)$

no max

(h) (4 points) Determine the intervals where $f(x)$ is concave up or concave down.

$$f'(x)=\frac{-4x}{(x-2)^3}$$

$$f''(x)=\frac{-4(x-2)^3+4x\cdot 3(x-2)^2}{(x-2)^6}$$

$$=\frac{(x-2)^2\left[-4x+8+12x\right]}{(x-2)^6}$$

$$=\frac{8(1+x)}{(x-2)^4}$$

$-1$ $2$

$-$ $+$ $+$

$\frown$ $\smile$ $\smile$

concave dn $(-\infty,-1)$

up $(-1,2)\cup(2,\infty)$

(i) (2 points) Determine the $x$ and $y$ coordinates of any points of inflection.

$$f(-1)=\cancel{\frac{-4(-1)}{(-1-2)^3}}\ \frac{(-1)^2}{(-1-2)^2}=\frac{1}{9}$$

$\left(-1,\frac{1}{9}\right)$

(j) (6 points) Based on your work above, sketch the graph of $f(x)$. Label all intercepts, extrema, and points of inflection. Label all horizontal and vertical asymptotes.