MAT 1575
Fall 2014
Professor K. Poirier
Test #3
December 2

Name (Print): Solutions.

Time Limit: 100 Minutes

This exam contains 6 pages and 9 problems, including one extra-credit problem. Check to see if any pages are missing. Print your name on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may use a calculator on this test. No other aids are allowed. Show all your work for full credit.

Total: 50 points, including 2 points for overall style (this includes using all notation correctly and presenting the work logically).

STYLE:

CHECK ALL ANTIDERIVATIVES BY DIFFERENTIATING

STATE EXPLICITLY ANY TESTS FOR CONVERGENCE OR DIVERGENCE THAT YOU USE

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1. (5 points) Determine whether the **sequence** converges or diverges. If it converges, determine which value it converges to. If it diverges, explain why.

$$a_n = \frac{3n^2 + 2n - 2}{3n^2 - 3}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3n^2+2n-2}{3n^2-3}$$

$$= \frac{3}{3}$$

$$= 1$$

So the sequence converges to 1

(Note: this question asked about the sequence not the series. If you'd been asked whether the series $\sum \frac{3n^2+2n-2}{3n^2-3}$ converged or diverged, you could have used the work above to show that the series diverges since the individual terms do not tend to 0).

2. (5 points) Determine whether the series converges or diverges. State which convergence test you are using.

$$\sum_{n=1}^{\infty} \frac{2+4^n}{6^n} = 2\sum \left(\tfrac{1}{6}\right)^n + \sum \left(\tfrac{4}{6}\right)^n$$

$\sum_{n=1}^{\infty} \left(\frac{1}{6}\right)^n$ is a geometric series which converges since $|r| = \frac{1}{6} < 1$
it converges to $\frac{1}{1-\frac{1}{6}} = \frac{6}{5}$

$\sum_{n=1}^{\infty} \left(\frac{4}{6}\right)^n$ is also geometric, $|r| = \frac{4}{6} = \frac{2}{3} < 1$ so it converges to
$\frac{1}{1-\frac{2}{3}} = \frac{3}{1}$

Therefore $\sum_{n=1}^{\infty} \frac{2+4^n}{6^n}$ converges to $2\left(\frac{6}{5}\right) + 3 = \frac{17}{5}$

3. (5 points) Determine whether the series converges or diverges. State which convergence test you are using.

$$\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2+2}}$$

(note that $\frac{n}{\sqrt{n^2+2}} < \frac{n}{\sqrt{n^2}} = 1$
but this is not a helpful direct comparison since
$\sum \frac{n}{\sqrt{n^2}} = \sum 1$ diverges.)

This gives us a hint, though .... as n gets very large n^2+2 is close to n^2, so it's quite possible that $\sum \frac{n}{\sqrt{n^2+2}}$ behaves like $\sum \frac{n}{\sqrt{n^2}}$ for large n.

Consider $\lim_{n\to\infty} \frac{n}{\sqrt{n^2+2}} = \lim_{n\to\infty} \frac{\frac{n}{n}}{\frac{1}{n}\sqrt{n^2+2}}$

$= \lim_{n\to\infty} \frac{\frac{n}{n}}{\sqrt{\frac{n^2+2}{n^2}}}$

$= \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{2}{n^2}}} = 1$

so the individual terms of $\sum \frac{n}{\sqrt{n^2+2}}$ do not approach 0, so $\sum \frac{n}{\sqrt{n^2+2}}$ can't converge (by the divergence test)

(the integral test also works here)

4. (5 points) Determine whether the series converges or diverges. State which convergence test you are using.

$$\sum_{n=1}^{\infty} \frac{3}{n! + 3^n}$$

you have some choices here, but notice that the denominator is getting very very large as $n \to \infty$, so a decent guess would be that the series should converge

$$0 < \frac{3}{n!+3^n} < \frac{3}{3^n} = 3\left(\frac{1}{3}\right)^n$$

Since $\sum 3\left(\frac{1}{3}\right)^n$ is a geometric series with $|r| = \frac{1}{3} < 1$, it converges

By the comparison test, $\sum \frac{3}{n!+3^n} \leq \sum \frac{3}{3^n}$

so $\sum \frac{3}{n!\,3^n}$ also converges

5. (5 points) Determine whether the series converges or diverges. State which convergence test you are using.

$$\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^3+1}}$$

limit comparison test: let $b_n = \frac{n}{\sqrt{n^3}} = n^{1-\frac{3}{2}} = n^{-\frac{1}{2}} = \frac{1}{\sqrt{n}}$

so $\sum b_n = \sum \frac{1}{\sqrt{n}}$ diverges. (p-test)

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n}{\sqrt{n^3+1}}}{\frac{n}{\sqrt{n^3}}}$$

$$= \lim_{n\to\infty} \sqrt{\frac{n^3}{n^3+1}}$$

$$= 1$$

$$0 < 1 < \infty$$

so since $\sum b_n$ diverges, $\sum \frac{n}{\sqrt{n^3+1}}$ also diverges.

6. (5 points) Determine whether the series converges or diverges. State which convergence test you are using.

$$\sum_{n=1}^{\infty} \frac{\ln(n)}{n^2}$$

First note that $\ln(n) < \sqrt{n}$ for all $n > 0$ (you can see this by comparing graphs of $f(x) = \ln(x)$ and $g(x) = \sqrt{x}$ for $x > 0$)

Therefore $0 \le \frac{\ln(n)}{n^2} \le \frac{\sqrt{n}}{n^2} = n^{\frac{1}{2}-2} = n^{-\frac{3}{2}} = \frac{1}{n^{\frac{3}{2}}}$

and $\sum \frac{1}{n^{\frac{3}{2}}}$ converges.

Therefore, by the direct convergence test, $\sum \frac{\ln(n)}{n^2}$ also <u>converges</u>

7. (5 points) Determine whether the series converges or diverges. State which convergence test you are using.

$$\sum_{n=1}^{\infty} \frac{n^n}{e^n}$$

(intuition: for large values of n, n^n will be <u>way</u> larger than e^n; so the series should diverge)

$\lim_{n\to\infty} \frac{n^n}{e^n} > \lim_{n\to\infty} \frac{3^n}{e^n} = \lim_{n\to\infty} \left(\frac{3}{e}\right)^n$

$\sum \left(\frac{3}{e}\right)^n$ is a geometric series which <u>diverges</u> since $|r| = \frac{3}{e} > 1$

$\sum \frac{n^n}{e^n} > \sum \left(\frac{3}{e}\right)^n$ so $\sum \frac{n^n}{e^n}$ also <u>diverges</u> by the comparison test

(Note: in the second version of the test: $\sum \frac{e^n}{n^n}$ <u>converges</u> by direct comparison with the convergent geometric series $\sum \left(\frac{e}{3}\right)^n$)

(note: the <u>root test</u> also works here)

8. (5 points) Determine the radius of convergence and interval of convergence for the power series.

$\sum_{n=1}^{\infty} \frac{3}{n} x^n$ ← center $=0$ so series definitely converges for $x=0$

ratio test: $a_{n+1} = \frac{3}{n+1} x^{n+1}$ $\frac{a_{n+1}}{a_n} = \frac{\frac{3}{n+1} x^{n+1}}{\frac{3}{n} x^n}$

$a_n = \frac{3}{n} x^n$ $= \frac{n}{n+1} x$

so $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{n}{n+1} x \right| = |x| \cdot \lim_{n\to\infty} \frac{n}{n+1} = |x|$

so the series converges if $|x| < 1$

Therefore, the radius of convergence is 1.

The interval of convergence includes $(-1, 1)$, but we have to check convergence at the interval endpoints individually

$x=-1$: $\sum_{n=1}^{\infty} \frac{3}{n} (-1)^n$ is an alternating series

The sequence $a_n = \frac{3}{n}$ is positive, decreasing, $\lim_{n\to\infty} \frac{3}{n} = 0$

Therefore $\sum \frac{3}{n} (-1)^n$ converges by the Leibniz test.

$x = 1$: $\sum_{n=1}^{\infty} \frac{3}{n}$ diverges by the p-test $(p=1)$.

Therefore the interval of convergence is $[-1, 1)$.

9. (8 points) Determine the Taylor series for the function $f(x) = \frac{1}{x}$ centered at $a = 1$. Determine the radius and interval of convergence for this Taylor series.

You have a few choices for how to approach this. Here's the approach that might be the most straightforward.

$f(x) = x^{-1}$	$f(1) = \frac{1}{1} = 1$	$\frac{f(1)}{0!} = 1$	$\frac{f(1)}{0!}(x-1)^0 = 1$
$f'(x) = -x^{-2}$	$f'(1) = \frac{-1}{1^2} = -1$	$\frac{f'(1)}{1!} = -1$	$\frac{f'(1)}{1!}(x-1)^1 = -(x-1)$
$f''(x) = (-2)(-1)x^{-3}$	$f''(1) = \frac{2}{1^3} = 2$	$\frac{f''(1)}{2!} = 1$	$\frac{f''(1)}{2!}(x-1)^2 = (x-1)^2$
$f'''(x) = (-3)(-2)(-1)x^{-4}$	$f'''(1) = \frac{-3\cdot 2}{1^4} = -6$	$\frac{f'''(1)}{3!} = -1$	$\frac{f'''(1)}{3!}(x-1)^3 = -(x-1)^3$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$f^{(n)}(x) = (-1)^n n!\, x^{-(n+1)}$	$f^{(n)}(1) = (-1)^n \cdot n!$	$\frac{f^{(n)}(1)}{n!} = (-1)^n$	$\frac{f^{(n)}(1)}{n!}(x-1)^n = (-1)^n(x-1)^n$

so $$T(x) = \sum_{n=0}^{\infty} (-1)^n (x-1)^n$$

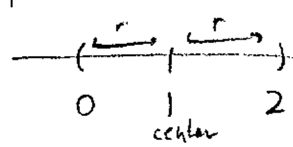
$$a_{n+1} = (-1)^{n+1}(x-1)^{n+1} \qquad \frac{a_{n+1}}{a_n} = \frac{(-1)^{n+1}(x-1)^{n+1}}{(-1)^n (x-1)^n}$$

$$a_n = (-1)^n (x-1)^n$$

so $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = |x-1|$

so $T(x)$ converges if $|x-1| < 1$ & radius of convergence $= 1$

check endpoints of



$(0,2)$ individually

$x = 0$: $T(0) = \sum (-1)^n (-1)^n$
$= \sum (-1)^{2n}$
$= \sum 1$
diverges

$x = 2$: $T(2) = \sum (-1)^n (1)^n$
$= \sum (-1)^n$
diverges

interval of convergence: $(0,2)$