MAT 1575
Fall 2014
Professor K. Poirier
Test #2
October 28

Name (Print): SOLUTIONS

Time Limit: 100 Minutes

This exam contains 6 pages and 8 problems, including one extra-credit problem. Check to see if any pages are missing. Print your name on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may use a calculator on this test. No other aids are allowed. Show all your work for full credit.

Total: 50 points, including 2 points for overall style (this includes using all notation correctly and presenting the work logically).

CHECK ALL ANTIDERIVATIVES BY DIFFERENTIATING

---

1. (6 points) Evaluate the following integral.

$$\int \frac{dx}{(x-2)(x-4)}$$

$$\frac{1}{(x-2)(x-4)} = \frac{A}{x-2} + \frac{B}{x-4} \quad \text{so} \quad 1 = A(x-4) + B(x-2)$$

$$= \frac{A(x-4) + B(x-2)}{(x-2)(x-4)}$$

if $x = 4$:
$1 = B(4-2)$
$\frac{1}{2} = B$

if $x = 2$:
$1 = A(2-4)$
$-\frac{1}{2} = A$

so $$\int \frac{dx}{(x-2)(x-4)} = \int \frac{-\frac{1}{2}}{x-2}\,dx + \int \frac{\frac{1}{2}}{x-4}\,dx$$

$$= -\frac{1}{2}\ln|x-2| + \frac{1}{2}\ln|x-4| + C$$

$$= \ln\left(\sqrt{\left|\frac{x-4}{x-2}\right|}\right) + C.$$

CHECK (we'll assume $x-2>0$ and $x-4>0$ - we really should check the other cases too).

$$\frac{d}{dx}\left(-\frac{1}{2}\ln(x-2) + \frac{1}{2}(x-4) + c\right) = \frac{-1}{2(x-2)} + \frac{1}{2(x-4)}$$

$$= \frac{-(x-4) + (x-2)}{2(x-2)(x-4)} = \frac{\cancel{2}}{\cancel{2}(x-2)(x-4)} \checkmark$$

2. (6 points) Evaluate the following integral.

$$\int \frac{(2x^2+3x-1)dx}{(x-1)(x+1)^2}$$

$$\frac{2x^2+3x-1}{(x-1)(x+1)^2} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$= \frac{A(x+1)^2 + B(x-1)(x+1) + C(x-1)}{(x-1)(x+1)^2}$$

So $2x^2+3x-1 = A(x+1)^2 + B(x-1)(x+1) + C(x-1)$

If $x=-1$:

$$2-3-1 = C(-1-1)$$
$$-2 = -2C$$
$$1 = C$$

if $x=1$:

$$2+3-1 = A(1+1)^2$$
$$4 = 4A$$
$$1 = A$$

If $x=0$: this choice is somewhat random - it's just easy to work with

$$-1 = \overset{A}{1}(0+1)^2 + B(0-1)(0+1) + \overset{C}{1}(0-1)$$
$$-1 = 1 - B - 1$$
$$B = 1.$$

so $\displaystyle\int \frac{2x^2+3x-1}{(x-1)(x+1)^2}dx = \int \frac{1}{x-1}dx + \int \frac{1}{x+1}dx + \int \frac{1}{(x+1)^2}dx$ let $u=x+1$, $du=dx$

$$= \ln|x-1| + \ln|x+1| + \int u^{-2}du$$
$$= \ln|x-1| + \ln|x+1| - \frac{1}{u} + c$$
$$= \ln|x-1| + \ln|x+1| - \frac{1}{x+1} + c$$

CHECK assume $x-1>0$ and $x+1>0$)

$$\frac{d}{dx}\left(\ln|x-1| + \ln|x+1| - \frac{1}{x+1} + c\right) = \frac{1}{x-1} + \frac{1}{x+1} + \frac{1}{(x+1)^2} = \frac{(x+1)^2+(x-1)(x+1)+(x-1)}{(x-1)(x+1)^2}$$

$$= \frac{x^2+2x+1+x^2-1+x-1}{(x-1)(x+1)^2} = \frac{2x^2+3x-1}{(x-1)(x+1)^2}$$

3. (6 points) Evaluate the following integral.

$$\int \frac{5x^2dx}{x^2+25}$$

x^2+25 cannot be factored over $\mathbb{R}$, so this is not a partial fractions question

let $x = 5\tan(\theta)$ $\qquad x^2 = 25\tan^2(\theta)$

$dx = 5\sec^2(\theta)d\theta$ $\qquad x^2+25 = 25\tan^2(\theta)+25$

$$= 25(\tan^2(\theta)+1)$$

$$= 25\sec^2(\theta)$$

so $$\int \frac{5x^2}{x^2+25}dx = \int \frac{5\cdot \cancel{25}\tan^2(\theta)\cdot 5\,\cancel{\sec^2(\theta)}d\theta}{\cancel{25}\,\cancel{\sec^2(\theta)}}$$

$$=25\int \tan^2(\theta)\,d\theta$$

$$=25\int (\sec^2(\theta)-1)\,d\theta$$

$$=25(\tan(\theta)-\theta)+c$$

$$=5x - 25\tan^{-1}\left(\frac{x}{5}\right)+c$$

since $x = 5\tan(\theta)$

$\frac{x}{5} = \tan(\theta)$

$\tan^{-1}\left(\frac{x}{5}\right) = \theta$

and $5x = 25\tan(\theta)$

CHECK:

$$\frac{d}{dx}\left(5x - 25\tan^{-1}\left(\frac{x}{5}\right)+c\right) = 5 - 25\frac{1}{\left(\frac{x}{5}\right)^2+1}\cdot\frac{1}{5}$$

$$= 5 - 5\left(\frac{1}{\frac{x^2}{25}+1}\right)$$

$$= 5 - 5\left(\frac{1}{\frac{x^2+25}{25}}\right)$$

$$= 5 - 5\left(\frac{25}{x^2+25}\right)$$

$$= \frac{5(x^2+25) - 5\cdot 25}{x^2+25}$$

$$= \frac{5x^2}{x^2+25} \checkmark$$

18

4. (6 points) Determine whether the integral converges or diverges. If it converges, evaluate it. If it diverges, explain why.

Integral is improper because one of the limits of integration is ∞

$$\int_0^\infty \frac{5dx}{(x+5)^4}$$

This integral will converge because - $4>1$
- the function has no vertical asymptotes in $[0,\infty)$.

$$= \lim_{R\to\infty} \int_0^R \frac{5}{(x+5)^4}\,dx$$

$$= \lim_{R\to\infty} -\frac{5}{3(x+5)^3}\Bigg]_0^R$$

CHECK: $\frac{d}{dx}\left(\frac{-5}{3(x+5)^3}\right) = \frac{d}{dx}\left(-\frac{5}{3}(x+5)^{-3}\right)$
$= \frac{-5}{3}(-3)(x+5)^{-4}$
$= 5(x+5)^{-4}$
$= \frac{5}{(x+5)^4}$ ✓

$$= \lim_{R\to\infty}\left(\frac{-5}{3(R+5)^3} - \frac{-5}{3(0+5)^3}\right)$$

$$= 0 + \frac{\cancel{5}}{3\cdot 5^{3\,2}}$$

$$= \frac{1}{75}$$

5. (6 points) Determine whether the integral converges or diverges. If it converges, evaluate it. If it diverges, explain why.

$$\int_0^{25} \frac{5dx}{(x-5)^4}$$

Integral is improper because:
$f(x) = \frac{5}{(x-5)^4}$ has a vertical asymptote at $x=5$, which is in $[0, 2\pi]$

The behavior of $f(x)$ near $x=5$ is similar to the behavior of $g(x)=\frac{1}{x^4}$ near its vertical asymptote near $x=0$, so it looks like the integral should diverge since $4>1$. (This is not precise, but gives us an idea of what we're looking for)

$$= \int_0^5 \frac{5dx}{(x-5)^4} + \int_5^{25} \frac{5dx}{(x-5)^4}$$

$$= \lim_{R\to 5^-}\int_0^R \frac{5dx}{(x-5)^4} + \lim_{R\to 5^+}\int_R^{25}\frac{5dx}{(x-5)^4}$$

(antiderivative check: same as in # 4)

$$= \lim_{R\to 5^-} \frac{-5}{3(x+5)^4}\Bigg]_0^R + \lim_{R\to 5^+}\frac{-5}{3(x+5)^4}\Bigg]_R^{25}$$

$$= \lim_{R\to 5^-}\left(\frac{-5}{3(R+5)^4} - \frac{-5}{3(0+5)^4}\right) + \lim_{R\to 5^+}\left(\frac{-5}{3(25+5)^4} - \frac{-5}{3(R+5)^4}\right)$$

not a finite number - so integral diverges

28 min

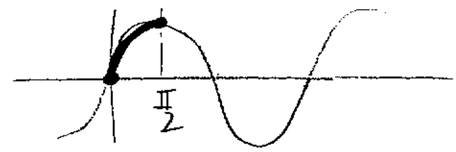
integral is improper because $\frac{1}{x\sin(x)}$ is unbounded on $[0, \frac{\pi}{2}]$

6. (6 points) Determine whether the integral converges or diverges. If it converges, evaluate it. If it diverges, explain why.

$$\int_0^{\frac{\pi}{2}} \frac{dx}{x\sin(x)}$$

it might be hard to tell just by looking at it whether the integral converges or diverges, so let's see if there's a helpful comparison

On $[0, \frac{\pi}{2}]$ $\quad 0 \le \sin(x) \le 1$



so $\quad \frac{1}{\sin(x)} \ge 1$

and $\quad \frac{1}{x\sin(x)} \ge \frac{1}{x} > 0$

but $\int_0^{\frac{\pi}{2}} \frac{1}{x} dx$ diverges. since $\int_0^{\frac{\pi}{2}} \frac{1}{x\sin(x)} dx \ge \int_0^{\frac{\pi}{2}} \frac{1}{x} dx$,

by comparison, $\int_0^{\frac{\pi}{2}} \frac{1}{x\sin(x)} dx$ also diverges.

32

7. (6 points) Determine whether the integral converges or diverges. If it converges, evaluate it. If it diverges, explain why.

integral is improper because one of the limits of integration is ∞

$$\int_1^{\infty} e^{-x} dx$$

$$= \lim_{R\to\infty} \int_1^R e^{-x} dx$$

CHECK: $\frac{d}{dx}(-e^{-x})$
$= -e^{-x}(-1)$
$= e^{-x}$ ✓

$$= \lim_{R\to\infty} -e^{-x} \Big]_1^R$$

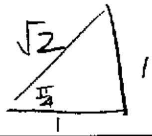
$$= \lim_{R\to\infty} \left(-e^{-R} - (-e^{-1})\right)$$

$$= \lim_{R\to\infty} \frac{-1}{e^R} + \frac{1}{e}$$

$$= 0 + \frac{1}{e}$$

$$= \frac{1}{e}$$

← this is a finite number, so the integral converges



8. (6 points) Find the Taylor polynomial of degree 3 for $f(x) = \cos(x)$ at $a = \frac{\pi}{4}$ (Do not use your calculator.)

$$f(x) = \cos(x) \qquad f\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

$$f'(x) = -\sin(x) \qquad f'\left(\frac{\pi}{4}\right) = -\sin\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$$

$$f''(x) = -\cos(x) \qquad f''\left(\frac{\pi}{4}\right) = -\cos\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$$

$$f'''(x) = \sin(x) \qquad f'''\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

$$T_3(x) = f\left(\frac{\pi}{4}\right) + f'\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right) + \frac{f''\left(\frac{\pi}{4}\right)}{2!}\left(x - \frac{\pi}{4}\right)^2 + \frac{f'''\left(\frac{\pi}{4}\right)}{3!}\left(x - \frac{\pi}{4}\right)^3$$

$$= \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\left(x - \frac{\pi}{4}\right) - \frac{1}{2\sqrt{2}}\left(x - \frac{\pi}{4}\right)^2 + \frac{1}{6\sqrt{2}}\left(x - \frac{\pi}{4}\right)^3$$

40