MAT 1475H
Spring 2014
Professor K. Poirier
Test #1
March 12, 2014

Name (Print): PROF P.

Time Limit: 100 Minutes

This exam contains 10 pages and 9 problems. Check to see if any pages are missing. Print your name on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may use a calculator on this test. No other aids are allowed.

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1. Evaluate the following limits $\lim_{x\to a} f(x)$. For each, use your answer to describe the behavior of the function $f(x)$ near the point $x = a$. (That just means use words to describe the graph.)

   (a) (5 points)

$$\lim_{x\to 0} \frac{4^{2x}-1}{4^x-1} \qquad \frac{4^0-1}{4^0-1} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{\cancel{(4^x-1)}(4^x+1)}{\cancel{4^x-1}}$$

$$= \lim_{x\to 0} 4^x+1$$

$$= 1+1$$

$$= 2$$

The graph of $f(x) = \frac{4^{2x}-1}{4^x-1}$ has a removable discontinuity at $(0,2)$.

   (b) (5 points)

$$\lim_{x\to 0} \frac{\sin(5x)}{\sin(3x)}$$

$$= \lim_{x\to 0} \frac{5\,\frac{\sin(5x)}{5x}}{3\,\frac{\sin(3x)}{3x}}$$

$$= \frac{5}{3} \frac{\lim_{5x\to 0} \frac{\sin(5x)}{5x}}{\lim_{3x\to 0} \frac{\sin(3x)}{3x}}$$

$$= \frac{5}{3}$$

The graph of $f(x) = \frac{\sin(5x)}{\sin(3x)}$ has a removable discontinuity at $(0, \frac{5}{3})$.

2.17

(c) (5 points)

$$\frac{4}{4^2 - 4 \cdot 4} \quad \frac{6}{0}.$$

$$\lim_{x\to 4^+} \frac{x}{x^2 - 4x}$$

$$= \lim_{x\to 4^+} \frac{x}{x(x-4)}$$

$$= \lim_{x\to 4^+} \frac{1}{x-4}$$

$$= \infty$$

The graph of $f(x) = \dfrac{x}{x^2 - 4x}$ has a vertical asymptote (from the right) at $x = 4$

(d) (5 points)

$$\lim_{x\to -\infty} \frac{x}{x^2 - 4x}$$

$= 0$ since degree of numerator is less than the degree of the denominator

The graph of $f(x)$ has a horizontal asymptote at $y=0$ (on the right)

4.17

2. (a) (5 points) Use the limit definition of the derivative to differentiate:

$$f(x) = x^3 - x^2$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^3 - (x+h)^2 - (x^3 - x^2)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 - \cancel{x^2} - 2xh - h^2 - \cancel{x^3} + \cancel{x^2}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(3x^2 + 3xh + h^2 - 2x - h)}{\cancel{h}}$$

$$= \lim_{h \to 0} 3x^2 + 3xh + h^2 - 2x - h$$

$$= 3x^2 - 2x.$$

(b) (5 points) Determine the equation of the line tangent to the graph of $f(x) = x^3 - x^2$ at the point where $x = -1$.

by (a), $f'(x) = 3x^2 - 2x$

so $f'(-1) = 3(-1)^2 - 2(-1)$

$= 3 + 2$

$= 5$

$f(-1) = (-1)^3 - (-1)^2$

$= -1 - 1$

$= -2.$

eqn of tan line:

$$\underset{\substack{\uparrow \\ f(-1)}}{y} - \underset{\substack{\uparrow \\ f(-1)}}{y_1} = m(x - \underset{\substack{\uparrow \\ -1}}{x_1})$$

$y - (-2) = 5(x - (-1))$

$y = 5x + 5 - 2$

$y = 5x + 3.$

7.46

3. Let $f(x) = x^5 + x^4 + x^3 + x^2 + x + 1$.

(a) (3 points) Find $f'(x)$.

$f'(x) = 5x^4 + 4x^3 + 3x^2 + 2x + 1$

(b) (2 points) Is $f(x)$ the derivative of another function $g(x)$? If so, find $g(x)$ so that $g'(x) = f(x)$.

Yes: $g(x) = \frac{1}{6}x^6 + \frac{1}{5}x^5 + \frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + C$

Check: $g'(x) = x^5 + x^4 + x^3 + x^2 + x + 1$

any constant

8.50

4. (10 points) Differentiate; do not simplify your answer"

$$f(x) = \left(\frac{e^{4x}+x}{\tan(x)+2^x}\right)^5 \cdot \sqrt{\frac{\ln(x)+x^2}{\sin^{-1}(3x)-1}}$$

Hint: This is a long one. It might be easier to break this down into smaller pieces, differentiate those, and then assemble the pieces to form your answer. Partial credit will be awarded for the individual pieces—as long as it's clear what you're doing—even if they're not assembled correctly in the end.

$$\frac{d}{dx}\left(e^{4x}+x\right) = 4e^{4x}+1$$

$$\frac{d}{dx}\left(\tan(x)+2^x\right) = \sec^2(x) + \ln(e)\cdot 2^x$$

$$\frac{d}{dx}\left(\ln(x)+x^2\right) = \frac{1}{x} + 2x$$

$$\frac{d}{dx}\left(\sin^{-1}(3x)-1\right) = \frac{1}{\sqrt{1-x^2}}$$

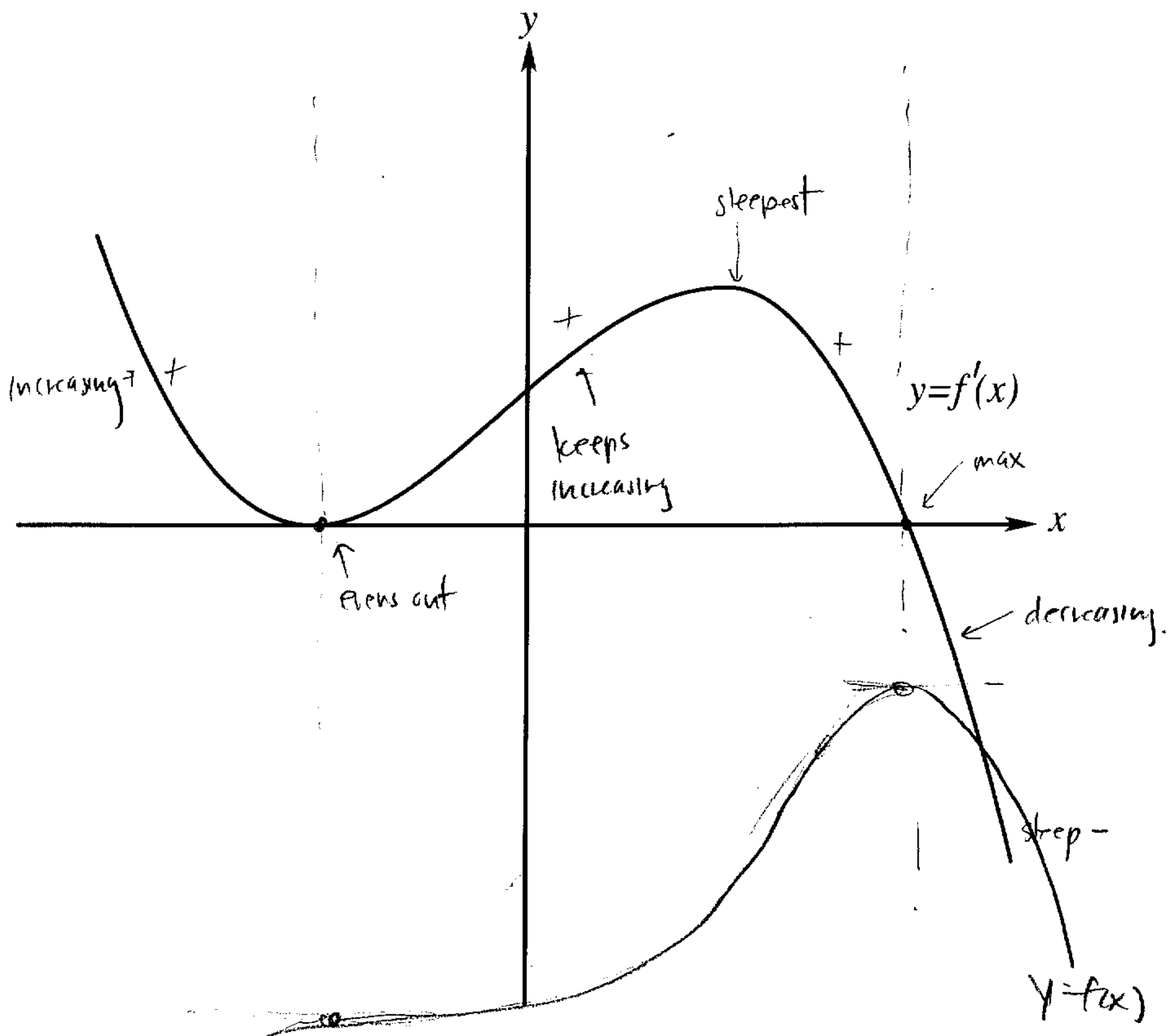
$$\frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}$$

$$f'(x) = 5\left(\frac{e^{4x}+x}{\tan(x)+2^x}\right)^4 \left(\frac{(4e^{4x}+1)(\tan(x)+2^x) - (e^{4x}+x)(\sec^2(x)+\ln(2)2^x)}{(\tan(x)+2^x)^2}\right) \cdot \sqrt{\frac{\ln(x)+x^2}{\sin^{-1}(3x)-1}}$$

$$+ \left(\frac{e^{4x}+x}{\tan(x)+2^x}\right)^5 \cdot \frac{1}{2}\left(\frac{\ln(x)+x^2}{\sin^{-1}(3x)-1}\right)^{-\frac{1}{2}} \cdot \left(\frac{(\frac{1}{x}+2x)(\sin^{-1}(3x)-1) - (\ln(x)+x^2)(\frac{1}{\sqrt{1-(3x)^2}}\cdot 3)}{(\sin^{-1}(3x)-1)^2}\right).$$

14.09

5. (a) (3 points) Use the following graph of $f'(x)$ to graph a function $f(x)$.



(b) (2 points) Is there more than one possible graph for $f(x)$? Explain.

Yes - I can add any constant number to f(x) - which raises or lowers the graph - without changing f'(x).

6.

$$f(x) = \begin{cases} x^3 \sin(\frac{1}{x}), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

(a) (3 points) Is $f(x)$ continuous at $x = 0$? Explain.

We have: $f(0) = 0$, so we need to see whether $\lim_{x\to 0} f(x)$ is also 0.

$\sin(\frac{1}{x})$ oscillates wildly near $x=0$ (so you might think that $\lim_{x\to 0} f(x)$ may not exist). but notice:

$-1 \le \sin(\frac{1}{x}) \le 1$ for all $x \neq 0$

so $-x^3 \le x^3 \sin(\frac{1}{x}) \le x^3$ — " —

Also: $\lim_{x\to 0}(-x^3) = 0 = \lim_{x\to 0} x^3$

so we can apply the squeeze theorem to see that $\lim_{x\to 0} x^3 \sin(\frac{1}{x}) = 0$

since this $= f(0)$,

$f(x)$ <u>is</u> continuous at $x=0$.

(b) (2 points) Is $f(x)$ differentiable at $x = 0$? If so, determine $f'(0)$.

If $x \neq 0$

$f'(x) = 3x^2 \sin(\frac{1}{x}) + x^3 \cos(\frac{1}{x}) \cdot (-\frac{1}{x^2})$ (product rule and chain rule)

$= 3x^2 \sin(\frac{1}{x}) - x\cos(\frac{1}{x})$

Let's see if $\lim_{x\to 0} f'(x)$ exists:

- $\lim_{x\to 0} 3x^2 \sin(\frac{1}{x}) = 0$ ← by an argument using the squeeze theorem - similar to (a)
- $\lim_{x\to 0} x\cos(\frac{1}{x}) = 0$

so $\lim_{x\to 0} f'(x)$ exists!

since $f(x)$ is continuous at $x=0$

<u>and</u> $\lim_{x\to 0} f'(x)$ exists,

then $f'(0)$ exists — that is $f(x)$ <u>is</u> differentiable at $x=0$

7. (10 points) Graph a single function $f(x)$ satisfying all of the following conditions.

(a) The domain of $f(x)$ is $\mathbb{R}$. ← graph must have y-values for all x-values

(b) $f(0) = 1$ ← graph has a point at (0,1)

(c) $\lim_{x\to 0^-} f(x) = -1$ ← graph has a hole at (0,-1), approaches it from left

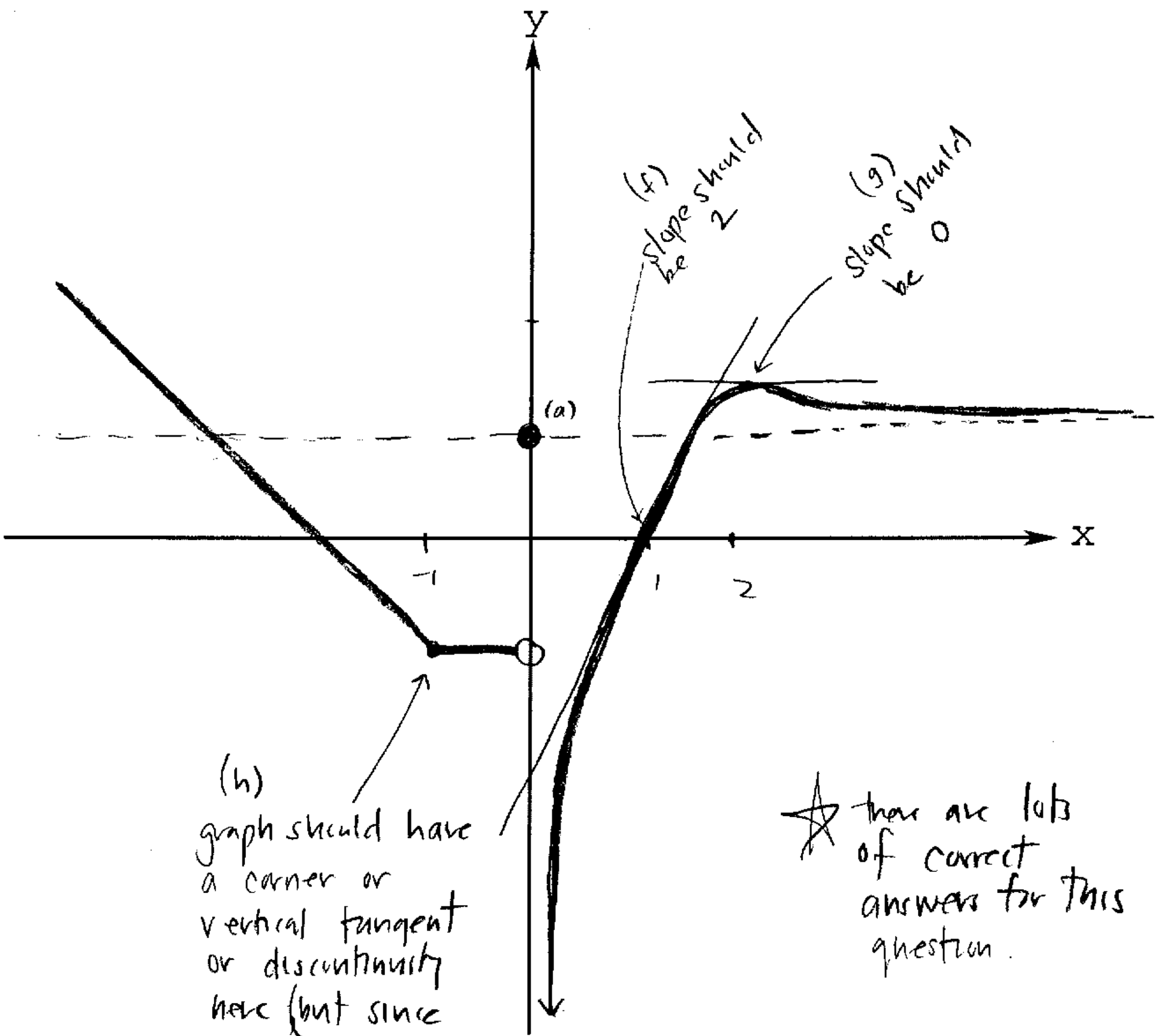
(d) $\lim_{x\to 0^+} f(x) = -\infty$ ← graph has a vertical asymptote at x=0, approaches $-\infty$ from the right of 0.

(e) $\lim_{x\to\infty} = 1$ ← graph has a horizontal asymptote at y=1 as x approaches ∞

(f) $f'(1) = 2$

(g) $f'(2) = 0$

(h) $f(x)$ is not differentiable at $x = -1$.



(h) graph should have a corner or vertical tangent or discontinuity here (but since domain is $\mathbb{R}$, there should be a y-coordinate for x-coordinate -1)

☆ there are lots of correct answers for this question.

8. Determine whether each of the following statements are true or false. You do not need to justify your answer.

   (a) (1 point) If $\lim_{x\to a} f(x)$ exists, then $f(x)$ is continuous at $x = a$.

   Circle one: true false (f(a) need not be defined)

   (b) (1 point) If $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ exists, then $f(x)$ is differentiable at $x = a$.

   Circle one: true false (f(a) need not be defined)

   (c) (1 point) If $f(x)$ is differentiable at $x = a$, then $f(x)$ is continuous at $x = a$.

   Circle one: true false (we saw this in class)

   (d) (1 point) If $\lim_{x\to a} f(x)$ exists, then $\lim_{x\to a^+} f(x)$ exists.

   Circle one: true false (the 2 sided limit exists if and only if the 2 one-sided limits exist and are =).

   (e) (1 point) If $f(a)$ exists, then $\lim_{x\to a} f(x)$ exists.

   Circle one: true false

   (e.g. $f(x) = \begin{cases} 0, & x \geq 0 \\ -1, & x < 0 \end{cases}$

   $f(0)=0$

   $\lim_{x\to 0} f(x)$ DNE)

24%

9. EXTRA CREDIT (This question will be graded stringently.)

(a) (1 point) State the ε-δ definition of limit

$$\lim_{x \to a} f(x) = L.$$

$\lim_{x \to a} f(x) = L$ if for all $\varepsilon > 0$ $\exists$ $\delta > 0$ st

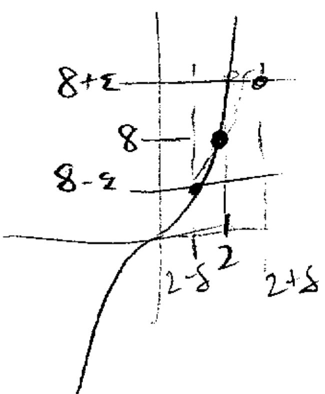
if $0 < |x-a| < \delta$

then $|f(x) - L| < \varepsilon$.

(b) (4 points) Use the $\varepsilon - \delta$ definition of limit to show

$$\lim_{x \to 2} x^3 = 8.$$

(we need to find a good δ).



want to find δ so that.

$\begin{cases} \text{if } x \in (2-\delta, 2+\delta) \\ \text{then } f(x) \in (8-\varepsilon, 8+\varepsilon) \end{cases}$

$$|f(x) - 8| = |x^3 - 2^3| = \underbrace{|x-2|}_{\delta}\,|x^2 + 2x + 4| < \varepsilon$$

assume $|x-2| < 1$

So $-1 < x - 2 < 1$

so $1 < x < 3$

If $x < 3$ then

$\cdot\ x^2 < 9$

$\cdot\ 2x < 6$

So $|x^2 + 2x + 4| < 9 + 6 + 4 = 19$.

and $|f(x) - 2| < |x-2| \cdot 19$

If $|x-2| < 1$ and $|x-2| < \frac{\varepsilon}{19}$

then $|f(x) - 8| < \frac{\varepsilon}{19} \cdot 19 = \varepsilon$

so if $\delta <$ the smaller of 1 and $\frac{\varepsilon}{19}$, then if $0 < |x-2| < \delta$, $|f(x) - 8| < \varepsilon$