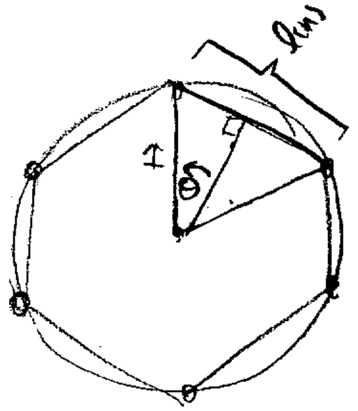
MAT1475H HOMEWORK #4 SOLUTIONS

2.7 #43

(a) As the number of sides of an inscribed polygon grows, the polygon looks more and more like the circle, so we expect

$$\lim_{n\to\infty} P(n) = 2\pi \quad \text{, the perimeter of the unit circle.}$$

(b)



(The author doesn't say so, but he's assuming the polygon is regular — that means that all its edges have the same length, and all the angles between the edges are also equal)

Let $\ell(n)$ be the length of an edge of the n-gon, so $P(n) = n\cdot \ell(n)$. From the right triangle in the figure above, $\frac{\ell(n)}{2} = \sin(\theta)$. $\theta = \frac{2\pi}{2n} = \frac{\pi}{n}$.

$$\text{and} \quad \frac{\ell(n)}{2} = \sin\left(\frac{\pi}{n}\right)$$

$$\text{so} \quad \ell(n) = 2\sin\left(\frac{\pi}{n}\right)$$

$$\text{and} \quad P(n) = 2n\sin\left(\frac{\pi}{n}\right)$$

(c) $\frac{n}{\pi}\sin\left(\frac{\pi}{n}\right) = \frac{P(n)}{2\pi}$ by (b)

$$\text{so} \lim_{n\to\infty} \frac{n}{\pi}\sin\left(\frac{\pi}{n}\right) = \lim_{n\to\infty}\frac{P(n)}{2\pi} = \frac{2\pi}{2\pi} \text{ by (a)}$$

$$= 1$$

(d) Let θ be as in the figure above in (b)

$$\lim_{n\to\infty} \theta = \lim_{n\to\infty} \frac{\pi}{n} = 0$$

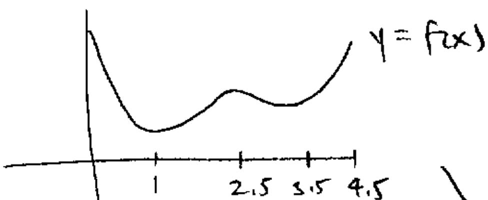
so

$$\lim_{\theta\to 0} \frac{\sin(\theta)}{\theta} = \lim_{n\to\infty} \frac{\sin(\frac{\pi}{n})}{\frac{\pi}{n}}$$

$$= \lim_{n\to\infty} \frac{n}{\pi} \sin\left(\frac{\pi}{n}\right)$$

$= 1$ by (c).

$\boxed{3.1\ \#49}$



$f'(x)$ is positive if the slope of the tangent line is positive – if the graph is increasing

$f(x)$ is increasing on the intervals $(1, 2.5)$ and $(3.5, 4.5)$

$\boxed{3.2\ \#43}$

(A) $\leftrightarrow$ (III)
(B) $\leftrightarrow$ (I)
(C) $\leftrightarrow$ (II)
(D) $\leftrightarrow$ (III)

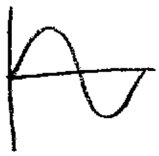
$\boxed{3.2\ \#44}$

- $f(x)$ is always positive
- $f(x)$ is decreasing, then increasing
- $f(x)$ has a horizontal tangent at $x=0$

- $g(x)$ is negative, then positive
- $g(x)$ is always increasing
- $g(x)$ has an x-intercept at $x=0$

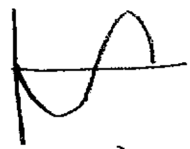
$$f'(x) = g(x)$$

3.2 # 45



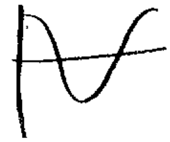
(A)

f(x)



(B)

$g'(x) = h(x)$



(C)

$f'(x) = g(x)$

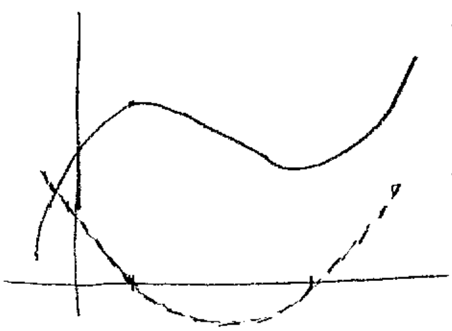
3.2 #66

(A) $\leftrightarrow$ (III)

(B) $\leftrightarrow$ (I)

(C) $\leftrightarrow$ (II)

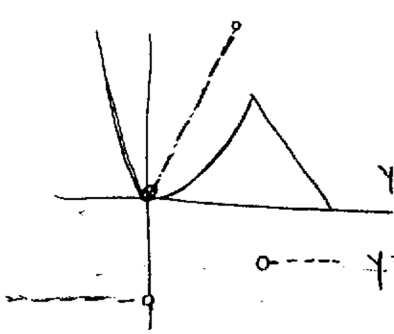
3.2 # 67



$y = f(x)$

$y = f'(x)$

3.2 #68

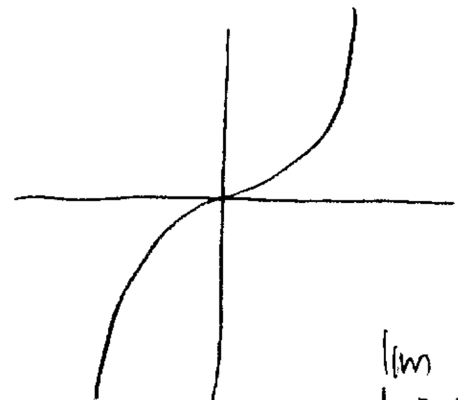


$y = f(x)$

o---- $y = f'(x)$

$\boxed{3.2 \# 69}$

$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$ so $f(x) = x|x| = \begin{cases} x^2 & \text{if } x \geq 0 \\ -x^2 & \text{if } x < 0 \end{cases}$



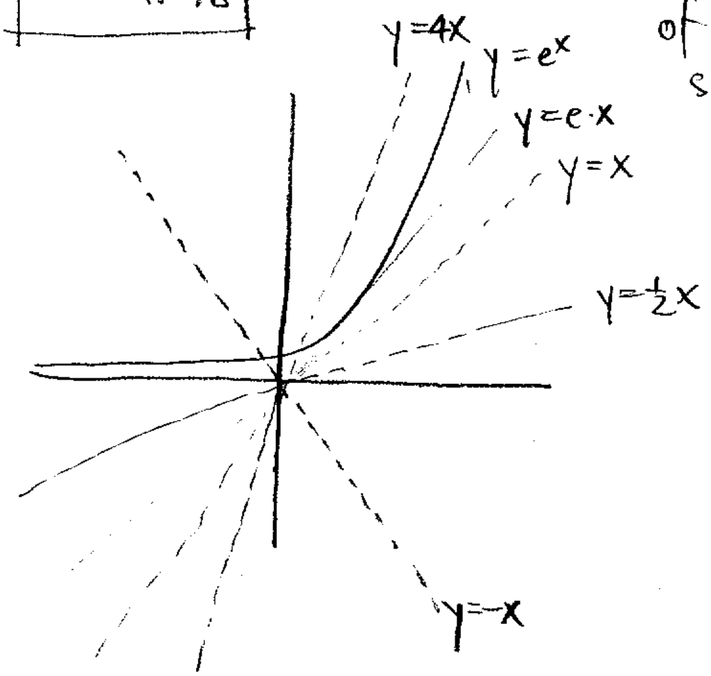
$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$\lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^+} \frac{h^2 - 0}{h}$$
$$= \lim_{h \to 0^+} h$$
$$= 0$$

$$\lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{-h^2 - 0}{h}$$
$$= \lim_{h \to 0^-} -h$$
$$= 0$$

so $f'(0) = 0$

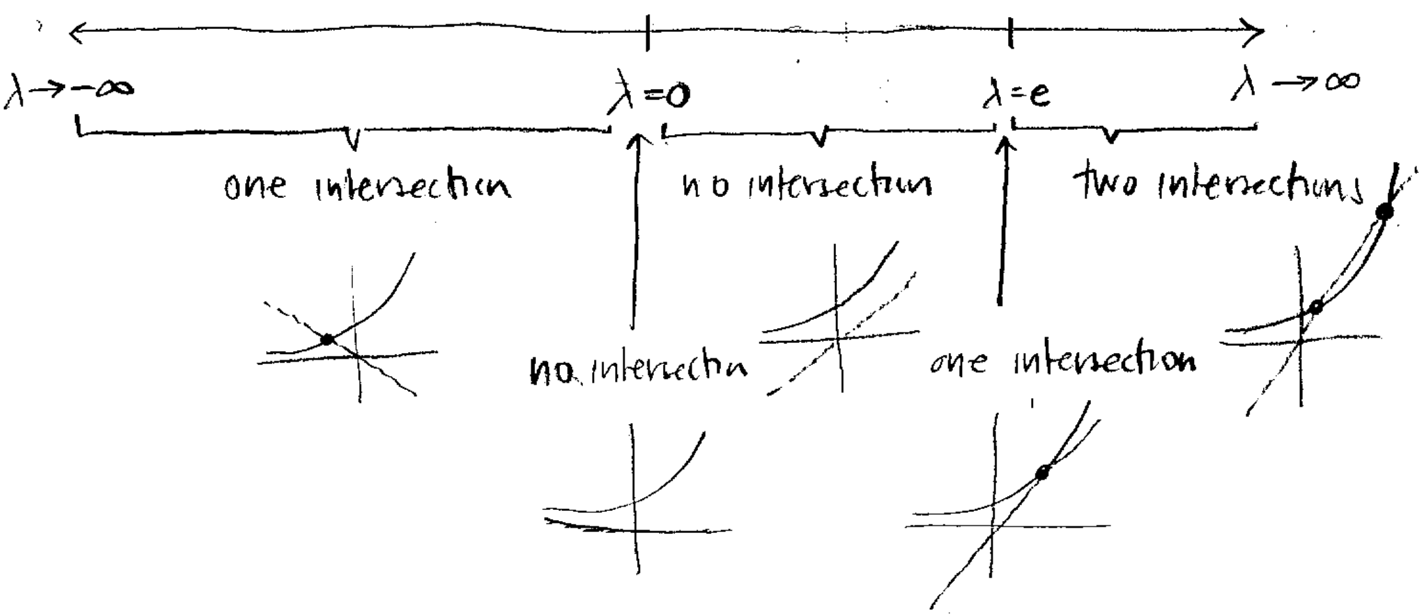
$\boxed{3.2 \# 98}$



Lines through the origin with a variety of slopes are graphed here. Imagine starting with a line $y = \lambda x$ where $\lambda < 0$ and $|\lambda|$ is large. Then imagine λ getting closer to 0, and becoming positive and growing so $\lambda > 0$ and $|\lambda|$ is large. This has the effect of rotating the line counter clockwise.

(next page)

as λ increases, $y = \lambda x$ rotates counter clockwise $\rightarrow$



We see that there is a unique solution to $e^x = \lambda x$ for the values of λ where $y = \lambda x$ intersects $y = e^x$ exactly once — for $\lambda \in (-\infty, 0)$ or $\lambda = e$