MAT 1475
Fall 2014
Professor K. Poirier
Test #1
October 6, 2014

Name (Print): SOLUTIONS

Time Limit: 100 Minutes

This exam contains 7 pages and 10 problems. Check to see if any pages are missing. Print your name on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may use a calculator on this test. No other aids are allowed.

Total: 50 points, including 2 style points overall for clear, precise, effective, and complete communication and justification (this includes correct use of all symbols and notation)

STYLE:

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1. (5 points) A ball dropped from a state of rest at time $t = 0$ travels a distance of $s(t) = 4.9t^2$ meters in t seconds. Compute the average velocity of the ball over the time interval $[1, 2]$.

$$\text{average velocity} = \frac{\Delta s}{\Delta t} = \frac{s(2) - s(1)}{2-1} \frac{m}{s}$$

$$= \frac{4.9 \cdot 2^2 - 4.9 \cdot 1^2}{1} \frac{m}{s}$$

$$= 14.7 \text{ m/s}$$

2. (5 points) Evaluate:

$$\lim_{x \to -1} \frac{x^3 + 4x}{x} = \frac{(-1)^3 + 4(-1)}{-1}$$

$$= \frac{-1-4}{-1}$$

$$= \frac{-5}{-1}$$

$$= 5$$

3. (5 points) Evaluate:

$$\lim_{x\to 1} \frac{x^2-1}{x-1}$$

$$= \lim_{x\to 1} \frac{(x-1)(x+1)}{x-1}$$

$$= \lim_{x\to 1} x+1$$

$$= 1+1$$

$$= 2$$

4. (5 points) Evaluate:

$$\lim_{x\to 0} \frac{3^{2x}-1}{3^x-1}$$

$$= \lim_{x\to 0} \frac{(3^x-1)(3^x+1)}{3^x-1}$$

$$= \lim_{x\to 0} 3^x+1$$

$$= 3^0+1$$

$$= 1+1$$

$$= 2$$

OR

$$\lim_{x\to 0} \frac{3^{2x}-1}{3^x-1}$$

$$= \lim_{x\to 0} \frac{3^{2x}-1}{3^x-1} \cdot \frac{3^x+1}{3^x+1}$$

$$= \lim_{x\to 0} \frac{(\cancel{3^{2x}-1})(3^x+1)}{\cancel{3^{2x}-1}}$$

$$= \lim_{x\to 0} 3^x+1 = 3^0+1 = 1+1 = 2$$

5. (5 points) Evaluate:

$$\lim_{x\to -1^+} \frac{x-1}{x^2-1}$$

Naïve substitution yields $\frac{-2}{0}$, which is of type $\frac{\text{nonzero}}{0}$, so the one-sided limit is either ∞ or $-\infty$.
To determine the sign, either substitute a value for x which is close to, but greater than, $x=-1$ or graph $f(x)=\frac{x-1}{x^2-1}$. Since $f(-0.9)$ is positive; $\lim_{x\to 1^+} \frac{x-1}{x^2-1} = \infty$

6. (5 points) Evaluate:

$$\lim_{x\to 0} \frac{\sin(7x)}{3x}$$

$$= \frac{1}{3} \lim_{x\to 0} \frac{\sin(7x)}{x}$$

$$= \frac{1}{3} \lim_{x\to 0} \frac{7\sin(7x)}{7x}$$

$$= \frac{7}{3} \lim_{7x\to 0} \frac{\sin(7x)}{7x}$$

$$= \frac{7}{3} \cdot 1$$

$$= \frac{7}{3}$$

7. (5 points) (a) Determine all points of discontinuity.

(b) State the type of discontinuity (removable, jump, infinite, or none of these).

$$f(x) = \frac{x-3}{x^2-x-6}$$

$$= \frac{x-3}{(x+2)(x-3)}$$

(a) $f(x)$ has discontinuities at $x=-2$ and $x=3$

(b) • $\lim_{x\to 3} \frac{x-3}{x^2-x-6} = \lim_{x\to 3} \frac{x-3}{(x+2)(x-3)}$

$= \lim_{x\to 3} \frac{1}{x+2}$

$= \frac{1}{3+2}$

$= \frac{1}{5}$ exists, so $x=3$ is a removable discontinuity for $f(x)$ (there is a hole in the graph at $(3, \frac{1}{5})$)

• $\lim_{x\to -2} \frac{x-3}{x^2-x-6}$: Naive substitution yields $\frac{-5}{0}$, so we consider the two one-sided limits separately (see solution for #5)

• $\lim_{x\to -2^-} \frac{x-3}{x^2-x-6}$ is either ∞ or $-\infty$

$f(-2.1)$ is negative, so $\lim_{x\to -2^-} \frac{x-3}{x^2-x-6} = -\infty$

• $\lim_{x\to -2^+} \frac{x-3}{x^2-x-6}$ is either ∞ or $-\infty$

$f(-1.9)$ is positive, so $\lim_{x\to -2^+} \frac{x-3}{x^2-x-6} = \infty$

So $x=-2$ is an infinite discontinuity for $f(x)$

(The graph has a vertical asymptote at $x=-2$)

8. (5 points) Use the limit definition of the derivative to determine the equation of the line tangent to the graph of $f(x) = x - x^2$ at the point where $x = -1$.

$$f(-1+h) = (-1+h) - (-1+h)^2$$
$$= -1+h-(1-2h+h^2)$$
$$= -1+h-1+2h-h^2$$
$$= -2+3h-h^2$$

$$f(-1) = (-1)-(-1)^2$$
$$= -1-1$$
$$= -2$$

Slope of tangent line:

$$m = f'(-1) = \lim_{h\to 0} \frac{f(-1+h)-f(-1)}{h}$$
$$= \lim_{h\to 0} \frac{-2+3h-h^2-(-2)}{h}$$
$$= \lim_{h\to 0} \frac{-\cancel{2}+3h-h^2+\cancel{2}}{h}$$
$$= \lim_{h\to 0} \frac{\cancel{h}(3-h)}{\cancel{h}}$$
$$= \lim_{h\to 0} 3-h$$
$$= 3-0$$
$$= 3$$

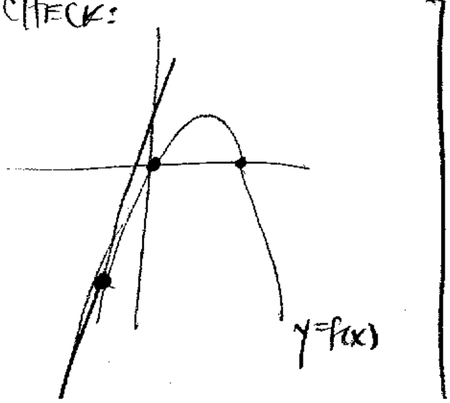
point of tangency $= (-1, f(-1))$
$= (-1,-2)$

equation of tangent line: $y - y_1 = m(x - x_1)$

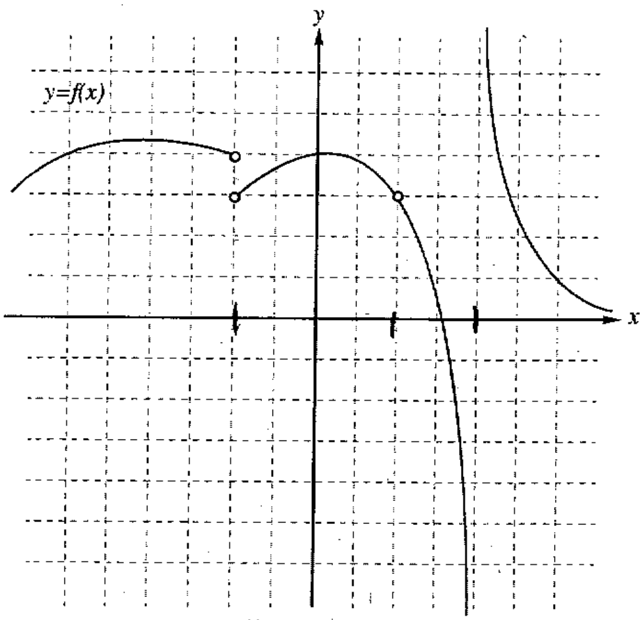
$$\begin{array}{ccc} y_1 & m & x_1 \\ \uparrow & \uparrow & \uparrow \\ -2 & 3 & -1 \end{array}$$

$$y-(-2) = 3(x-(-1))$$
$$y+2 = 3(x+1)$$
$$y = 3x+3-2$$
$$y = 3x+1$$

CHECK:



9. (5 points) Determine the two one-sided limits at the following x-values for the function $f(x)$ shown in the figure. and state whether the two-sided limit exists at each of these points.



(a) $x = 2$

Limit from the left: $\lim_{x\to 2^-} f(x) = 3$

Limit from the right: $\lim_{x\to 2^+} f(x) = 3$

Two-sided limit: $\lim_{x\to 2} f(x) = 3$

(b) $x = 4$

Limit from the left: $\lim_{x\to 4^-} f(x) = -\infty$

Limit from the right: $\lim_{x\to 4^+} f(x) = \infty$

Two-sided limit: $\lim_{x\to 4} f(x)$ does not exist

(c) $x = -2$

Limit from the left: $\lim_{x\to -2^-} f(x) = 4$

Limit from the right: $\lim_{x\to -2^+} f(x) = 3$

Two-sided limit: $\lim_{x\to -2} f(x)$ does not exist

TRUE/FALSE QUESTIONS - YOU DO NOT NEED TO JUSTIFY YOUR ANSWER

10. Determine whether each of the following statements are true or false. You do not need to justify your answer.

   (a) (1 point) If $\lim_{x \to a} f(x)$ exists, then $f(x)$ is continuous at $x = a$.

   Circle one: true false

   (b) (1 point) If $\lim_{x \to a} f(x)$ exists, then $\lim_{x \to a^+} f(x)$ exists.
   Circle one: true false

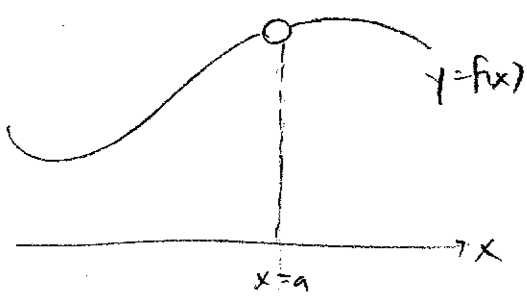
   If the two-sided limit exists, that means that the two one-sided limits exist and are equal, so in particular, the right-hand limit exists

   (c) (1 point) If $f(a)$ exists, then $\lim_{x \to a} f(x)$ exists.

   Circle one: true false

COUNTEREXAMPLES FOR (a) AND (c):

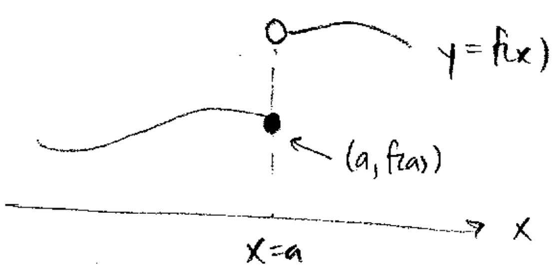
(a)



If $f(x)$ has a removable discontinuity at $x=a$, this means $\lim_{x \to a} f(x)$ exists, but $\lim_{x \to a} f(x) \neq f(a)$

So just because the limit exists doesn't mean the function is continuous

(c)



If $f(x)$ has a jump discontinuity at $x=a$ as in the graph, then $f(a)$ exists, but $\lim_{x \to a} f(x)$ does not since the two one-sided limits are not equal.