MAT1575 Module 7 – Alternating series.

Objectives: Approximate the value of π using an alternating series.

1. State the Alternating Series Test.
2. Consider the alternating series

$$\sum_{n=0}^{\infty}(-1)^n a_n = a_0 - a_1 + a_2 - a_3 + \cdots$$

where the a_n's are positive, DECREASING, and $\lim_{n\to\infty} a_n = 0$. We can write the alternating sum as

$$\sum_{n=0}^{\infty}(-1)^n a_n = \sum_{n=0}^{2k}(-1)^n a_n - a_{2k+1} + a_{2k+2} - a_{2k+3} + a_{2k+4} - a_{2k+5} + \cdots$$

where $2k$ is an even number. This means that

$$\sum_{n=0}^{\infty}(-1)^n a_n - \sum_{n=0}^{2k}(-1)^n a_n = -a_{2k+1} + a_{2k+2} - a_{2k+3} + a_{2k+4} - a_{2k+5} + \cdots$$

We will show that

$$-a_{2k+1} \leq -a_{2k+1} + a_{2k+2} - a_{2k+3} + a_{2k+4} - a_{2k+5} + \cdots \leq 0$$

by grouping the terms in the sum in two different ways.

First,

$$\begin{aligned} -a_{2k+1} + a_{2k+2} - a_{2k+3} + a_{2k+4} - a_{2k+5} + \cdots &= -a_{2k+1} + (a_{2k+2} - a_{2k+3}) + (a_{2k+4} - a_{2k+5}) + \cdots \\ &\geq -a_{2k+1} + 0 + 0 + \cdots \quad \text{(Why?)} \\ &= -a_{2k+1}. \end{aligned}$$

Second,

$$\begin{aligned} -a_{2k+1} + a_{2k+2} - a_{2k+3} + a_{2k+4} - \cdots &= (-a_{2k+1} + a_{2k+2}) + (-a_{2k+3} + a_{2k+4}) + \cdots \\ &\leq 0 + 0 + \cdots \\ &= 0. \end{aligned}$$

This shows that

$$\sum_{n=0}^{\infty}(-1)^n a_n - \sum_{n=0}^{2k}(-1)^n a_n \geq -a_{2k+1}.$$

Taking absolute values of both sides is like multiplying by -1 since all of the quantities are negative, so

$$\left|\sum_{n=0}^{\infty}(-1)^n a_n - \sum_{n=0}^{2k}(-1)^n a_n\right| \leq a_{2k+1}.$$

3. Repeat the above argument with the finite sum from $n = 0$ to $n = 2k - 1$ (instead of $2k$) to show that the partial sum of the alternating series from $n = 0$ to $n = 2k - 1$ differs (in absolute value) from the true value of the alternating series by at most a_{2k}.

4. The answers to questions 2 and 3 show that, for any $M \geq 1$,

$$\left|\sum_{n=0}^{\infty}(-1)^n a_n - \sum_{n=0}^{M-1}(-1)^n a_n\right| \leq \underline{\qquad}.$$

   In other words, the error is at most the size of the next term in the sum (in absolute value).

   NOTE: The work from questions 2 and 3 does not give a rigorous proof of 4, it simply outlines one way of explaining the error term for alternating series.

5. Use your answer to question 4 together with the known identity:

$$\pi = \sqrt{12}\sum_{n=0}^{\infty}\frac{(-1)^n}{3^n(2n+1)}$$

   to approximate π with an error $< 10^{-6}$ using Desmos.

6. What is the smallest number of terms you need to sum to approximate π with an error $< 10^{-11}$? DO NOT GUESS, use the work above to guarantee that your answer is right!

7. What is the smallest number of terms you need to sum to approximate π with an error $< 10^{-2}$ using the following known identity?

$$\pi = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$$

   Compare your answer to the previous question.