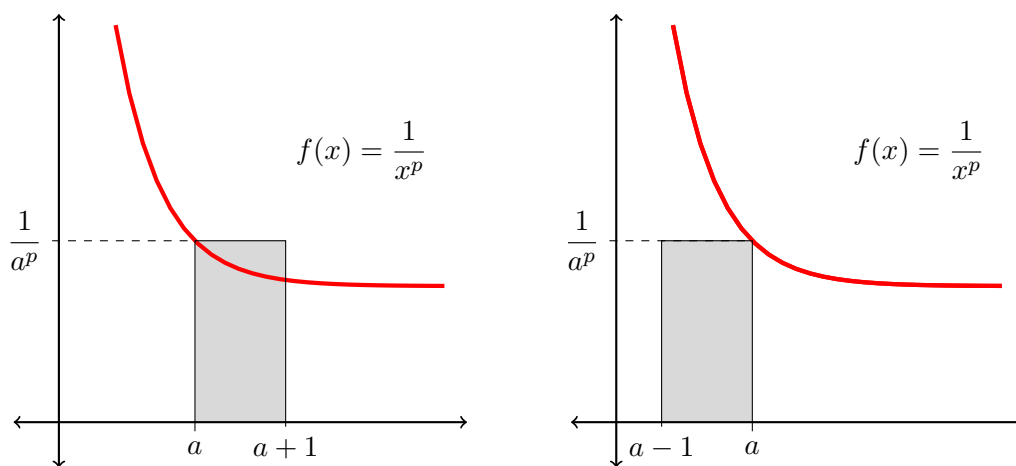


MAT1575 Module 6 – Comparing improper integrals and series.

Objectives: Construct inequalities for estimating the value of $\sum_{n=1}^{\infty} \frac{1}{n^p}$ when $p > 1$. Test your inequalities using Desmos.

1. Show that $\int_a^{\infty} \frac{1}{x^p} dx = \frac{1}{(p-1)a^{p-1}}$ for $p > 1$ and $a > 0$. (We will need this later.)
2. Write the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$, where $p > 1$, as a sum from $n = 1$ to $n = M$ plus a sum from $n = M + 1$ to infinity.



(a) Lower bound for sum in terms of an integral.

(b) Upper bound for sum in terms of an integral.

Figure 1: Bounding a sum in terms of an integral.

3. Compute a formula for the area of the shaded rectangle in Figure 1a and a formula (as an integral) for the area under the curve from a to $a + 1$. Which value is larger?
4. Compute a formula for the area of the shaded rectangle in Figure 1b and a formula (as an integral) for the area under the curve from $a - 1$ to a . Which value is larger?
5. Use your answer to question 3 to write a lower bound for $\sum_{n=M+1}^{\infty} \frac{1}{n^p}$ using an improper integral. Be careful with the bounds of integration!

6. Use your answer to question 4 to write an upper bound for $\sum_{n=M+1}^{\infty} \frac{1}{n^p}$ using an improper integral. Be careful with the bounds of integration!
7. Use your answers to questions 1, 2, 5, and 6 to write lower and upper bounds for $\sum_{n=1}^{\infty} \frac{1}{n^p}$ in terms of a finite sum from $n = 1$ to $n = M$ plus an improper integral. (Use your answer to 1 to evaluate the improper integrals.)
8. Test your inequalities using Desmos against the following examples:
- (a) Verify by approximation that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ using $M = 10, 20, 50$.
- (b) Verify by approximation that $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$ using $M = 10, 20, 50$.
- (c) Verify by approximation that $\sum_{n=1}^{\infty} \frac{1}{n^3} = 1.2020569031\dots$ using $M = 10, 20, 50$.