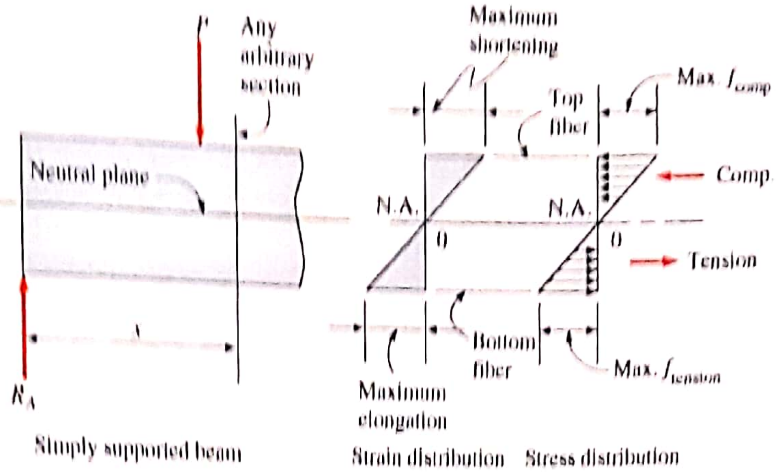


FIGURE 14.3 Stress and strain distribution.

at the neutral axis to a maximum compressive stress at the top outer fiber and a maximum tensile stress at the bottom outer fiber.

14.2 THE FLEXURE FORMULA

For the purpose of analysis and design of beams, it is necessary to work with the relationship between bending stresses, bending moment, and the geometric properties of a cross section. Whether we are dealing with analysis in which, perhaps, a stress is to be determined or with a design in which an allowable stress is used as a factor in the selection of a member, the same basic relationship, called the *flexure formula*, will apply. The derivation and application of the flexure formula requires that we know the location of the neutral (or centroidal) axis. In a symmetrical member, this is easily obtained by inspection. For unsymmetrical members, the location of the centroidal axis must be calculated in the manner described in Chapter 7.

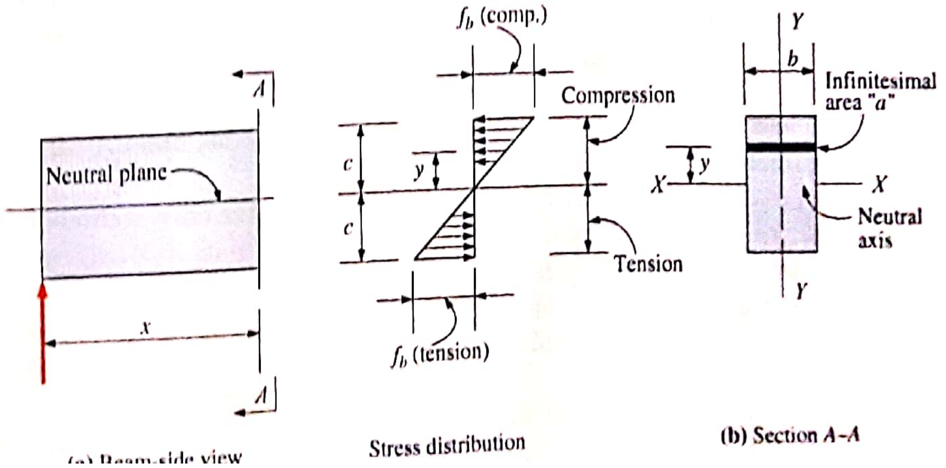
Figure 14.4a shows the side view of a small part of a simply supported beam with a typical stress distribution at some arbitrary location. Figure 14.4b shows the cross section of the beam. The beam is symmetrical with respect to the X-X and Y-Y axes. The section shown is rectangular, but the following discussion is valid for a cross section of any shape having a Y-Y axis of symmetry where the loading is applied in the plane of the Y-Y axis. The line XX is the neutral axis of the cross section. The width of the section is denoted b. The distance from the neutral axis to an infinitesimal area a is denoted y. The distance from the neutral axis to the outer fiber of the cross section is denoted c.

The bending stress (either tension or compression) that develops at the outer fiber (a distance c from the neutral axis) will be referred to, for now, as $f_{b(\max)}$. This maximum bending stress at the outer fiber is the bending stress that is usually of greatest importance. We can also calculate, by proportion, the bending stress that develops at a distance y from the neutral axis. For now, we refer to this lesser stress as f_b. It can be expressed as

$$f_b = \frac{f_{b(\max)}(y)}{c}$$

Recalling that force is equal to the product of stress and area, we write the force developed on infinitesimal area a as

$$\frac{f_{b(\max)}(y)}{c}(a)$$



The moment of the preceding force with respect to the X–X axis can be calculated as

$$\frac{f_{b(max)}}{c}\left(y^2\right)(a)$$

Finally, the total moment, with respect to the X–X axis, of all the internal forces acting on all the infinitesimal areas can be written as

$$\frac{f_{b(max)}}{c}\Sigma y^2(a)$$

As shown in Chapter 8, the mathematical quantity $(\Sigma y^2 a)$ is the moment of inertia of a cross section about its X–X axis and is represented by the symbol *I*. Therefore, the expression for the total moment may be written as

$$\frac{f_{b(max)}}{c} I$$

Since this total internal moment holds in equilibrium the moment due to the external loads *M*, it is sometimes called an *internal resisting moment* and can be expressed as

$$M = \frac{f_{b(max)} I}{c} \quad (14.1)$$

where
- M = the bending moment due to external loads, or the internal resisting moment (lb-in., k-ft) (N · m)
- $f_{b(max)}$ = the bending stress developed at the outer fiber (psi, ksi) (Pa)
- I = the moment of inertia about the neutral axis (in.^4) (m^4)
- c = the distance from the neutral axis to the outer fiber (in.) (m)

Rewriting this expression and solving for the stress gives

$$f_{b(max)} = \frac{Mc}{I} \quad (14.2)$$

where all terms are as previously defined.

Since stresses are proportional to distance from the neutral axis, we can also write the expression for bending stress developed at any distance *y* from the neutral axis:

$$f_b = \frac{My}{I} \quad (14.3)$$

where f_b in this case will be less than the maximum bending stress that occurs at the outer fiber. Note that substitution of *c* for *y* in Equation (14.3) results in Equation (14.2). Since, for all practical purposes, it is the maximum bending stress that is of importance, we omit the "(max)" from $f_{b(max)}$, with the understanding that it is the maximum bending stress with which we are working (unless otherwise noted).

We can also rewrite Equation (14.1) to find the maximum resisting moment, or allowable moment, for a cross section. To use this expression, the limiting value of bending stress, called the allowable bending stress, must be known:

$$M_R = \frac{F_b I}{c} \quad (14.4)$$

where
- M_R = the allowable moment or moment strength (lb-in., k-ft) (N · m)
- F_b = the alloable bending stress (psi, ksi) (Pa)

and *I* and *c* are as previously defined.

In these various forms of the flexure formula, note that the moment of inertia *I* and the distance *c* are both functions of the size and shape of the beam cross section. They are both geometric properties of the cross section and do not depend on the material or span length of the beam or on the type of loading on the beam. The quantity *I*/*c*, therefore, is also a geometric property. *I*/*c* is called the *section modulus* and is generally represented by the symbol *S*. The section modulus has units of in.^3 and can be calculated using the moment of inertia from Table 3 (inside the back cover) divided by the distance *c*.

The flexure formula can thus be rewritten and used in the following forms depending on whether the problem is one of analysis or design. For analysis problems,

$$f_b = \frac{M}{S} \quad (14.5)$$

or

$$M_R = F_b S \quad (14.6)$$

For design problems, for some materials, such as timber, the most convenient form is

$$\text{required } S = \frac{M}{F_b} \quad (14.7)$$

The values of the section modulus for standard rolled structural shapes and other standard shapes, both metallic and nonmetallic, can be found in various publications of the American Institute of Steel Construction, the American Forest & Paper Association, and other such organizations. Some of this material is provided in the appendices of this book.

As a physical analogy, the section modulus can be considered a measure of the comparative strength of beams. All other things being equal, if the section modulus of the cross section of beam *A* is twice as great as that of beam *B* of the same material, beam *A* will have double the bending strength.

Note that if the cross section of a beam is not symmetrical with respect to its X–X axis (the axis of bending), it will have two section modulus values of different magnitude since the *c* values for the top and bottom fibers will be different. Also, the centroidal axis will not be located at midheight of the cross section.

Since beams are such common members in structures and since the use of the flexure formula is basic to beam analysis and design, the limitations of the formula should be recognized. Its use is valid only under certain conditions, which may be briefly itemized as follows:

1. The beam must be straight before loading.
2. The beam must be homogeneous, obey Hooke's law, and have equal moduli of elasticity in both tension and compression.
3. The loads and reactions must lie in a vertical plane of symmetry perpendicular to the longitudinal axis of the beam.
4. The beam must be of uniform cross section.
5. The maximum bending stress must not exceed the proportional limit.
6. The beam must have adequate lateral buckling resistance.
7. All component parts of the beam must have adequate localized buckling resistance.
8. The beam must be relatively long in proportion to its depth.
9. The cross section must not be disproportionately wide.
10. The dimensional changes must not be appreciably affected by shear strains that are also present.

Despite the many limiting factors, the flexure formula may be applied quite satisfactorily in the design and analysis of most beams normally encountered.

14.3 COMPUTATION OF BENDING STRESSES

In the analysis of beams, one type of problem involves calculating maximum bending stress. This stress will occur at the outer fibers at the section of the beam where the bending moment is a maximum. In computing the maximum bending stress, the location and the magnitude of the maximum bending moment must be calculated first. The value of the section modulus (or the moment of inertia and the c distance) must then be calculated or obtained from standard tables of properties of sections. By substituting these values in Equation (14.5), or Equation (14.2), we obtain the maximum bending stress.

The following examples illustrate the use of the flexure formula. They are not to be interpreted as complete beam-analysis problems. Various types of more comprehensive problems follow in Section 14.7.

EXAMPLE 14.1 A nominal 8-in.-by-12-in. timber member, shown in Figure 14.5, is used as a beam that is subjected to a vertical loading. For maximum bending strength, the beam is oriented so that the 12-in. dimension is vertical. Calculate the section modulus of the beam with respect to the axis of bending (the X–X axis). Use dressed dimensions.

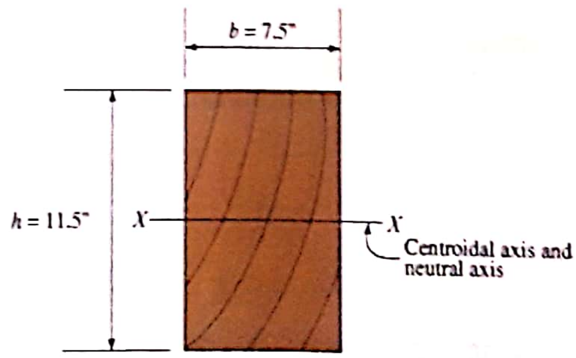


FIGURE 14.5 Beam cross section.

Solution The dressed dimensions for the timber member are $7\frac{1}{2}$ in. by $11\frac{1}{2}$ in. For solid rectangular shapes, an expression for section modulus can be derived as follows:

$$S = \frac{I}{c} = \frac{(bh^3/12)}{h/2} = \frac{bh^2}{6}$$

This convenient expression is used in dealing with solid homogeneous rectangular shapes, such as timber members.

Substitute numerical values:

$$S_x = \frac{bh^2}{6} = \frac{(7.5\text{ in.})(11.5\text{ in.})^2}{6} = 165.3\text{ in.}^3$$

This value may be verified in Appendix E.

EXAMPLE 14.2 The timber member of Example 14.1 is used as a simply supported beam on a 16 ft span carrying a uniformly distributed line load of 400 lb/ft, as shown in Figure 14.6. Calculate the maximum induced bending stress. Neglect the weight of the beam.

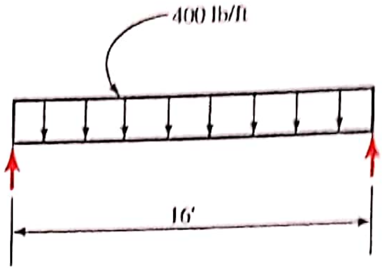


FIGURE 14.6 Load diagram.

Solution The maximum stress can be calculated using Equation (14.5). First, calculate the maximum bending moment. From Appendix H,

$$M = \frac{wL^2}{8} = \frac{(400\ \text{lb/ft})(16\ \text{ft})^2}{8} = 12{,}800\ \text{lb-ft}$$

Using the section modulus for the cross section from the previous example, substitute into Equation (14.5) to find the maximum bending stress:

$$f_b = \frac{M}{S} = \frac{(12{,}800\ \text{lb-ft})(12\text{in./ft})}{165.3\ \text{in.}^3} = 929\ \text{psi}$$

EXAMPLE 14.3 Assume that, due to a construction error, the beam of the previous two examples was placed incorrectly in the field and the small dimension ($7\frac{1}{2}$ in. dressed) was oriented vertically. For the same span length and loading, calculate the new maximum bending stress that would be developed.

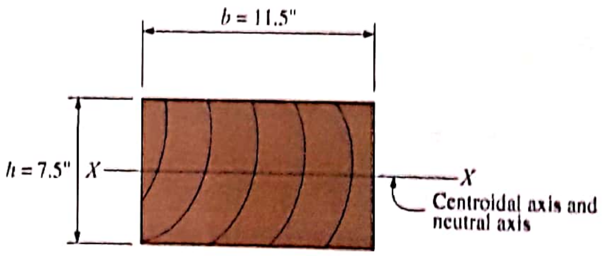


FIGURE 14.7 Beam cross section.

Solution The maximum bending moment remains the same at 12,800 lb-ft. The orientation of the cross section is shown in Figure 14.7. Note that the horizontal axis is designated as axis X–X. Calculate the section modulus:

$$S_x = \frac{bh^2}{6} = \frac{(11.5\ \text{in.})(7.5\ \text{in.})^2}{6} = 107.8\ \text{in.}^3$$

Calculate the maximum bending stress:

$$f_b = \frac{M}{S} = \frac{(12{,}800\ \text{lb-ft})(12\ \text{in./ft})}{107.8\ \text{in.}^3} = 1425\ \text{psi}$$

The results of Examples 14.2 and 14.3 indicate that the orientation of a beam cross section is an important strength consideration. For maximum bending resistance, the bending axis (perpendicular to the applied loads) should be the axis about which the section modulus is the largest value.

EXAMPLE 14.4 A W30 × 99 hot-rolled structural steel wide-flange shape is used as a simply supported beam on a span of 32 ft, as shown in Figure 14.8. The beam supports a superimposed uniformly distributed load of 4.0 kips/ft in addition to its own weight. Calculate the maximum bending stress.

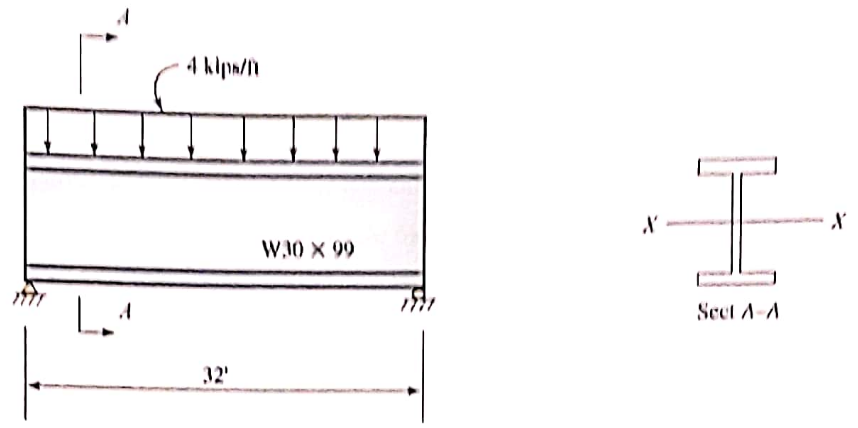


FIGURE 14.8 Beam for Example 14.4.

Solution First, calculate the maximum bending moment. The total load is the sum of the superimposed load (4.0 kips/ft) and the weight of the beam (0.099 kips/ft). From Appendix H,

$$M = \frac{wL^2}{8} = \frac{(4.10\ \text{k/ft})(32\ \text{ft})^2}{8} = 525\ \text{k-ft}$$

The section modulus can be obtained from Appendix A, which is a partial tabulation of the AISC tables of dimensions and properties of structural shapes. Assume that the shape is oriented so that bending is about the strong axis. Hence, $S = 269\ \text{in.}^3$

Calculate the maximum bending stress:

$$f_b = \frac{M}{S} = \frac{(525\ \text{k-ft})(12\ \text{in./ft})}{269\ \text{in.}^3} = 23.4\ \text{ksi}$$

EXAMPLE 14.5 An extra-strong steel pipe having a nominal diameter of 127 mm is to be used as a simple beam with a span length of 7 m, as shown in Figure 14.9. The pipe supports a concentrated load at midspan of 6 kN. Calculate the maximum bending stress due to (a) the weight of the pipe alone and (b) the concentrated load alone.

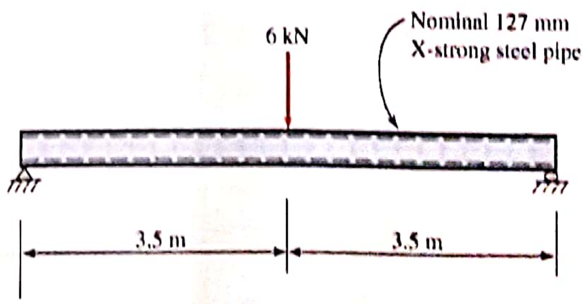


FIGURE 14.9 Beam for Example 14.5.

Solution From Appendix B (SI), the section modulus S is found to be $115 \times 10^3\ \text{mm}^3$. The weight of the pipe is calculated from

$$w = mg$$
$$= 30.9\ \text{kg/m}\ (9.81\ \text{m/sec}^2) = 303\ \text{N/m}$$

For a simply supported beam, the moment due to the beam's own weight is obtained from

$$M = \frac{wL^2}{8} = \frac{(303\ \text{N/m})(7\ \text{m})^2}{8}$$

$$= 1856\ \text{N}\cdot\text{m}$$

The maximum moment due to the applied concentrated load is

$$M = \frac{PL}{4} = \frac{(6\ \text{kN})(7\ \text{m})}{4} = 10.5\ \text{kN}\cdot\text{m}$$

The flexure formula of Equation (14.5) is then used to compute the bending stress, as follows:

(a) Due to the beam's own weight:

$$f_b = \frac{M}{S} = \frac{1856 \times 10^3\ \text{N}\cdot\text{mm}}{115 \times 10^3\ \text{mm}^3} = 16.14\ \text{MPa}$$

(b) Due to the applied concentrated load:

$$f_b = \frac{M}{S} = \frac{10.5 \times 10^6\ \text{N}\cdot\text{mm}}{115 \times 10^3\ \text{mm}^3} = 91.3\ \text{MPa}$$

We could also determine the total stress. Since the calculated stresses occur at the same point in the beam, they are additive:

$$\text{total } f_b = 16.14 + 91.3 = 107.4\ \text{MPa}$$