MAT 2440 Assignment #3 - The Lamplighter Group L_2

1 L_2 as a dynamical system (Notes from Assignment #2)

We take our definition of dynamical system to be an "object" along with a specific set of modifications that can be performed (dynamically) upon this object. In this case, the object is a bi-infinite straight road with a lamp post at every street corner and a marked lamp (the position of the lamplighter). There are two possible types of modifications: the lamplighter can walk any distance in either direction from a starting point and the lamplighter can turn the lamps "on" or "off." At any given moment the lamplighter is at a particular lamp post and a finite number of lamps are illuminated while the rest are not. We refer to such a moment, or configuration, as a "state" of the road (not to be confused with the "state" of an automaton). Any time the configuration changes, the road is in a new state. The road's state is changed over time by the lamplighter either walking to a different lamp post or turning lamps on or off (or both).

In Figure 1, the bi-infinite road is represented by a number line; the lamps are indexed by the integers. Lamps that are on are indicated by stars; lamps that are off by circles. The position of the lamplighter is indicated by an arrow pointing to an integer. The current state of the road is called the *lampstand*.

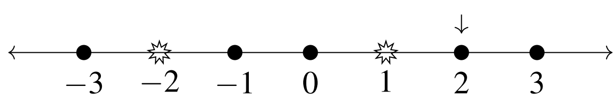

Figure 1: *A lampstand where two lamps are illuminated and the lamplighter stands at 2.*

Let us call the set of all possible lampstands $\mathscr{L}$. Now that we have a visual image, we can formalize the dynamics of changing a lampstand by specifying distinct tasks which the lamplighter can perform on any element of $\mathscr{L}$.

1. Move right to the next lamp.

2. Move left to the next lamp.

3. Switch the current lamp's status (from on to off or off to on).

4. Do nothing.

For any reconfiguration, the lamplighter performs only finitely many tasks. These tasks can be interpreted as functions τ, σ and I, whose domain and range are $\mathscr{L}$. Given a lampstand $l \in \mathscr{L}$, $\tau(l)$ is the result of performing the first task on l, $\sigma(l)$ is the result of performing the third task on l and $I(l)$ the result of performing the fourth task on l.

Proposition 1. σ is bijective.

Proof. To see that σ is onto, let l_1 be any lampstand in $\mathscr{L}$, and suppose that the lamplighter stands at lamp k. Define l_0 as the lampstand whose lamplighter stands at lamp k and whose lamps are in the same configuration as those in l_1, **except** for lamp k. If k is on in l_1, it is off in l_0; if it is off in l_1, it is on in l_0. Then $\sigma(l_0) = l_1$.

To see that σ is one-to-one, suppose that $\sigma(l_0) = \sigma(l_0') = l_1$, with the lamplighter in l_1 standing at lamp k. Since σ does not cause the lamplighter to move, the only effect it has on a lampstand is to switch the status of the current lamp. Whatever the status of lamp k is in l_1, it must be in the opposite state in both l_0 and l_0'. All other attributes of both l_0 and l_0' must match the other attributes of l_1; hence, $l_0 = l_0'$.
□ □

The reader will prove that τ is also bijective in Exercise 2 at the end of this chapter. Hence, both σ and τ have inverses. $\tau^{-1}(l)$ is the result of performing the second task on l. Note that σ is its own inverse. Thus $\sigma^2 = 1$.

If we let the lamplighter stand at 0 with all the lamps turned off, this configuration is called the *empty lampstand* and is denoted e. See Figure 2.

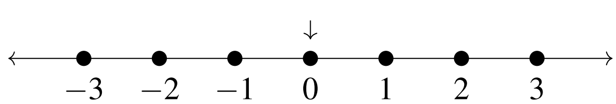

Figure 2: *The empty lampstand e*

Example 1. Consider the lampstand l_1 in Figure 3.

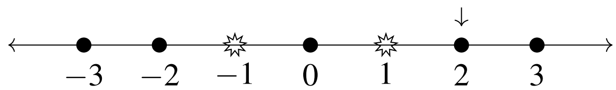

Figure 3: *The lampstand l_1*

Starting with the empty lampstand e, we can apply a composition of functions τ, τ^{-1}, σ and I to achieve l_1. For instance the composition $\tau\sigma\tau\tau\sigma\tau^{-1}$ (or $\tau \circ \sigma \circ \tau \circ \tau \circ \sigma \circ \tau^{-1}$) applied to e yields the lampstand configuration l_1. In keeping with standard function notation, the order of the composition is such that τ^{-1} is applied to e first and so on, reading from right to left. Figure 4 shows the details of the transformation from e to l_1.

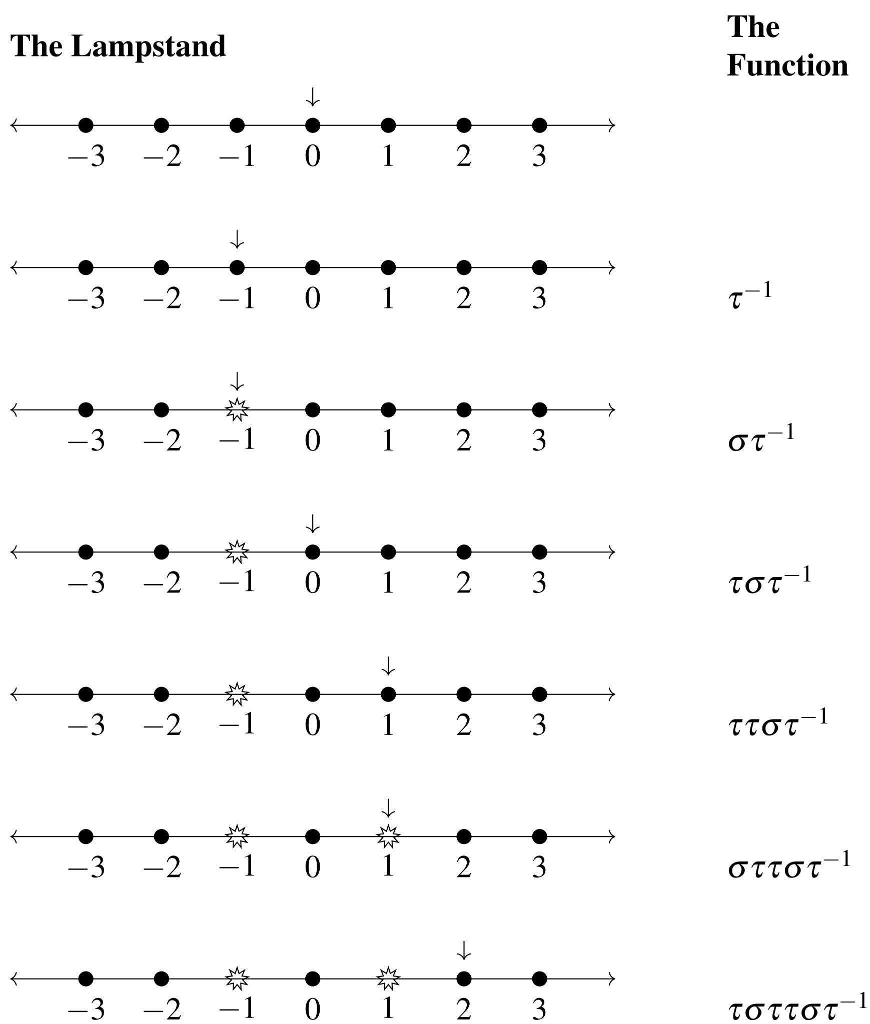

Figure 4: *A sequence of lampstands from the empty lampstand to* l_1 $\diamond$

To get the same lampstand l_1 (Figures 3 and 4), we could easily have applied a different function composition to e, for instance

$$\tau I \tau\tau I \sigma\tau^{-1}\tau^{-1}\sigma\tau.$$

For that matter, pick any $l \in \mathscr{L}$ as input. These two different-looking functions always have the same output.

$$\tau\sigma\tau\tau\sigma\tau^{-1}(l) = \tau I\tau\tau I\sigma\tau^{-1}\tau^{-1}\sigma\tau(l).$$

It doesn't matter that there are different function compositions representing the same lampstand, since two functions are defined to be the same function as long as the domains are the same and the outputs are the same. However, some function compositions are clearly "shorter" than others. Here "shorter" refers to the number of tasks in the function composition. This begs the question, is there a "shortest" function composition for a given lampstand configuration? You explored this in Assignment #2 earlier in the semester. We are now ready to define our lamplighter group L_2. Each element of L_2 is a particular **configuration** of the road (i.e., an element of $\mathscr{L}$); however, the lampstands can be identified (bijectively) with the set of all function compositions of σ, τ, and τ^{-1}, evaluated at the empty lampstand. Thus, τ is identified with $\tau(e)$, and σ is identified with $\sigma(e)$ (see Figure **??**). Rather than draw a picture, we will usually refer to each lampstand by identifying it with a function composed of the building blocks τ, τ^{-1} and σ. This allows us

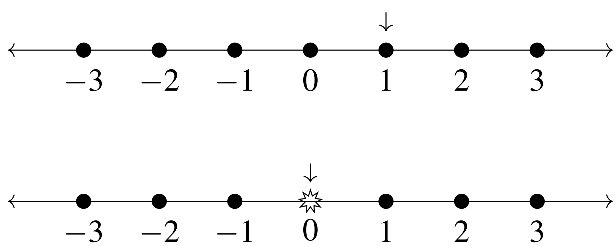

Figure 5: *The lampstands $\tau(e)$ (above) and $\sigma(e)$ (below)*

to define the identity element $I(e)$, which we will simply call "e" (since group elements are lampstands). We must also define the group multiplication. It is difficult to imagine "multiplying" two lampstands together; but thinking of our elements as functions, it is easy. The binary operation is function composition. If l_1 and l_2 are in L_2, then their product l_1l_2 is the composite function l_2 followed by l_1. Since the composition of bijective functions is also bijective, the group operation is well-defined, and inverses exist. Associativity follows since the group operation is function composition.

Example 2. Let $l_1 = \tau\sigma\tau^2\sigma\tau^{-1}$ and $l_2 = \tau\sigma\tau$; then

$$l_1l_2 = (\tau\sigma\tau^2\sigma\tau^{-1})(\tau\sigma\tau) = \tau\sigma\tau^3.$$

Looking at it dynamically, and working from right to left, we start with l_2: a lamplighter, whose name is Gilbert, starts at 0 on the lampstand and moves one step to the right (τ) to 1, turns on the lamp (σ), then moves one more step to the right (τ) to 2 and stops. Gilbert is now standing at 2 on the lampstand, which becomes the new home base as he performs the moves for l_1. For l_1, he moves one step to the left, from 2 to 1, and turns **off** the lamp, then moves two steps right to 3 and turns on the lamp before finally moving one step to the right, ending up at 4. Note that the same configuration is achieved if we use the reduced form of $l_1 l_2$, removing all pinches.

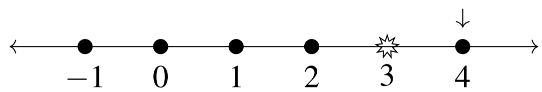

Figure 6: *The lampstand* $l_1 l_2$

◇

At this point we can see that L_2 forms a group. It has identity element e and is generated by τ and σ. Inverse elements are easy to find. For example, the inverse of the element $l_1 = \tau\sigma\tau^2\sigma\tau^{-1}$ is $l_1^{-1} = (\tau\sigma\tau^2\sigma\tau^{-1})^{-1} = \tau\sigma\tau^{-2}\sigma\tau^{-1}$ and its lampstand configuration is shown in Figure **??**. This completes the check that L_2 is a group.

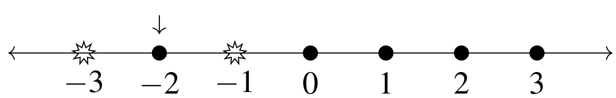

Figure 7: *The lampstand* l_1^{-1}

Given a particular lampstand, there is a visual method for finding its inverse without having to work out the dynamics of the configuration. For the lamplighter positioned at n, and a particular configuration of lighted lamps, reflect the lamplighter from n to $-n$, and translate the set of lighted lamps $-n$ units along the number line (compare Figure 3 with Figure **??**).

2 L_2 with ordered pair elements, using an infinite direct sum (Notes for Assignment #3)

Another way of representing the lampstand elements of L_2 rather than by functions is by using an ordered pair. The first entry represents the location of the lamplighter and the second entry makes use of an infinite sum construction to indicate which lamps are illuminated. This allows us to encode the important information of a particular lampstand configuration concisely.

The elements of L_2 can be represented by

$$\left\{ (n,\vec{x}) \mid n \in \mathbb{Z},\ \vec{x} \in \bigoplus_{i\in\mathbb{Z}} (\mathbb{Z}_2) \right\}$$

The $\vec{x}$ are infinite tuples in which each entry is assigned a value of 0 or 1. However, since only finitely many entries of the $\vec{x}$ can have value 1, we introduce *pointer* notation. $X = \{x_1, \ldots, x_p\}$ is a finite set of pointers corresponding to the positions of the entries in the infinite tuple whose value is 1. For example, $(12, \{-1, 3\})$ means the lamplighter is standing at 12 and the lamps at -1 and 3 are lit.

In order to add $\vec{x}$ to $\vec{y}$, we use the corresponding sets of pointers X and Y and calculate their *symmetric difference*, $X \triangle Y$. Since addition is taking place in $\mathbb{Z}_2$, if an integer k appears in both X and Y, indicating that the kth entry of both $\vec{x}$ and $\vec{y}$ is 1, the integer k drops out from the set $X \triangle Y$. Any integer that appears in exactly one of the pointer lists will appear in $X \triangle Y$ as well. For instance, if $X = \{-4, -1, 7, 12\}$ and $Y = \{-3, -1, 5, 7, 12\}$, then $X \triangle Y = \{-4, -3, 5\}$.

For two elements $l_1, l_2 \in L_2$, with $l_1 = (a, X)$ and $l_2 = (b, Y)$, the group operation is

$$l_1 \star l_2 = (\ a+b\ ,\ [X+b] \triangle Y\ ),$$

where the expression $X + b$ represents a new pointer set obtained by adding the integer b to each element of set X.

To illustrate, let us consider the elements g and h of L_2, whose lampstand configurations are given in Figures **??** and **??**.

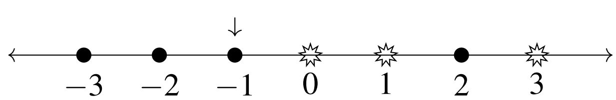

Figure 8: *The lampstand g*

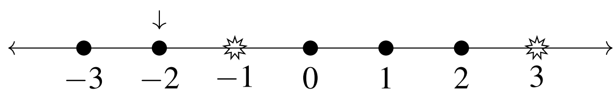

Figure 9: *The lampstand h*

Using the ordered pair notation, g corresponds to the ordered pair $(-1,\{0,1,3\})$ and h corresponds to the ordered pair $(-2,\{-1,3\})$. According to our formula,

$$\begin{aligned} g\star h &= (-1,\{0,1,3\})\star(-2,\{-1,3\}) \\ &= (-1+(-2)\,,\,[\{0,1,3\}+(-2)]\triangle\{-1,3\}) \\ &= (-3,\{-2,-1,1\}\triangle\{-1,3\}) \\ &= (-3,\ \{-2,1,3\}). \end{aligned}$$

You may wonder why the group multiplication involves a "shift" in the second component of the ordered pair representing g. Recall from Section **??** that g and h can be represented dynamically as a sequence of tasks performed on the empty lampstand, $\tau^{-1}\sigma\tau^{-1}\sigma\tau^{-2}\sigma\tau^{3}$ for g and $\tau^{-1}\sigma\tau^{-4}\sigma\tau^{3}$ for h. Once the tasks for h are performed on the empty lampstand, the lamplighter is standing at -2, which becomes the **new** home base as we perform the moves for g! To visualize this "shift" followed by addition (mod 2), consider Figure **??**, where the lighted lamps of the lampstand for h appear, followed by the lighted lamps of g shifted 2 units left, which is denoted as 'g-shift.' Once we "add (mod 2) straight down" and calculate the new position of the lamplighter, the result is $g\star h$.

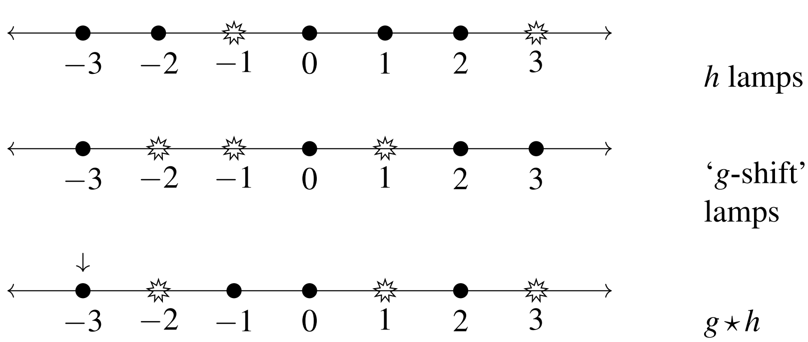

Figure 10: $g\star h$

Like the dynamical system, this representation describes lampstands; however, our elements are ordered pairs. The empty lampstand is represented by $e = (0, \emptyset)$. The generating elements in this description, corresponding to σ and τ, are $s = (0, \{0\})$ and $t = (1, \emptyset)$. The inverse of s is s, and $t^{-1} = (-1, \emptyset)$.

3 Assignment #3

This assignment is due on XX/XX/20XX at 10 am. You may submit it electronically as a pdf document or as a hard copy. Assignments late by 1 day will be penalized by 25%, 2 days late 50%, 3 days late 75% and any later they will no longer be accepted.

Please be sure this writing is your own - do NOT borrow from a friend - you must submit your own work. I want to hear your own voice, not read a copy and paste of some other source!!!

Exercise 1. Let $g = \tau^{-3}\sigma\tau\sigma$. [10 points each]

a. Find g^{-1}.

b. Draw g as an element of $\mathscr{L}$.

c. Draw g^{-1} as an element of $\mathscr{L}$.

d. Write g and g^{-1} as ordered pairs as described in Section 2.

Exercise 2. Given h corresponding to the ordered pair $(-2, \{-2, 3\})$ and g corresponding to the ordered pair $(-1, \{-1, 2\})$ [10 points each]

a. find $h \star g$ and

b. draw its lampstand configuration.

Exercise 3.

a. Write a general procedure for computing the product of any two lampstand elements in pseudocode. (Use the pseudocode format described in your textbook in Appendix 3.) [5 points]

b. Translate your pseudocode into Python by authoring a Trinket and **sharing it with me AND a partner**, also be sure to include a hard copy of your Python code with this assignment. [10 points]

c. Using the ‘print’ command to show your results, test that your code AND your group member’s code works correctly by checking that

   i. $gg^{-1} = (0, \emptyset)$, where $g = \tau^{-3}\sigma\tau\sigma$ from Exercise 1 [5 points]
   ii. $g^{-1}g = (0, \emptyset)$, where $g = \tau^{-3}\sigma\tau\sigma$ from Exercise 1 [5 points]
   iii. $h \star g$ matches your calculation from Exercise 2. [5 points]

d. Give your partner a feedback report including: the outcome of your testing and, if the test revealed the code is not working or accurate, what you think may have gone wrong and how it can be addressed. You must include a copy of this feedback report with your assignment in order to receive credit for this part. [10 points]