**MAT 2440 Assignment #2 - The Lamplighter Group $L_2$**

# 1 $L_2$ as a dynamical system

We take our definition of dynamical system to be an "object" along with a specific set of modifications that can be performed (dynamically) upon this object. In this case, the object is a bi-infinite straight road with a lamp post at every street corner. There are two possible types of modifications: the lamplighter can walk any distance in either direction from a starting point and the lamplighter can turn the lamps "on" or "off." At any given moment the lamplighter is at a particular lamp post and a finite number of lamps are illuminated while the rest are not. We refer to such a moment, or configuration, as a "state" of the road (not to be confused with the "state" of an automaton). Any time the configuration changes, the road is in a new state. The road's state is changed over time by the lamplighter either walking to a different lamp post or turning lamps on or off (or both).

In Figure 1, the bi-infinite road is represented by a number line; the lamps are indexed by the integers. Lamps that are on are indicated by stars; lamps that are off by circles. The position of the lamplighter is indicated by an arrow pointing to an integer. The current state of the road is called the *lampstand*.

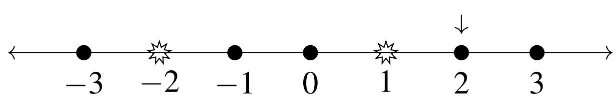


Figure 1: *A lampstand where two lamps are illuminated and the lamplighter stands at 2.*

Let us call the set of all possible lampstands $\mathscr{L}$. Now that we have a visual image, we can formalize the dynamics of changing a lampstand by specifying distinct tasks which the lamplighter can perform on any element of $\mathscr{L}$.

1. Move right to the next lamp.

2. Move left to the next lamp.

3. Switch the current lamp's status (from on to off or off to on).

4. Do nothing.

For any reconfiguration, the lamplighter performs only finitely many tasks. These tasks can be interpreted as functions $\tau$, $\sigma$ and $I$, whose domain and range are $\mathscr{L}$. Given a lampstand $l \in \mathscr{L}$, $\tau(l)$ is the result of performing the first task on $l$, $\sigma(l)$ is the result of performing the third task on $l$ and $I(l)$ the result of performing the fourth task on $l$.

**Proposition 1.** $\sigma$ is bijective.

*Proof.* To see that $\sigma$ is onto, let $l_1$ be any lampstand in $\mathscr{L}$, and suppose that the lamplighter stands at lamp $k$. Define $l_0$ as the lampstand whose lamplighter stands at lamp $k$ and whose lamps are in the same configuration as those in $l_1$, **except** for lamp $k$. If $k$ is on in $l_1$, it is off in $l_0$; if it is off in $l_1$, it is on in $l_0$. Then $\sigma(l_0) = l_1$.

To see that $\sigma$ is one-to-one, suppose that $\sigma(l_0) = \sigma(l_0') = l_1$, with the lamplighter in $l_1$ standing at lamp $k$. Since $\sigma$ does not cause the lamplighter to move, the only effect it has on a lampstand is to switch the status of the current lamp. Whatever the status of lamp $k$ is in $l_1$, it must be in the opposite state in both $l_0$ and $l_0'$. All other attributes of both $l_0$ and $l_0'$ must match the other attributes of $l_1$; hence, $l_0 = l_0'$.
$\square$ $\square$

The reader will prove that $\tau$ is also bijective in Exercise 2 at the end of this chapter. Hence, both $\sigma$ and $\tau$ have inverses. $\tau^{-1}(l)$ is the result of performing the second task on $l$. Note that $\sigma$ is its own inverse. Thus $\sigma^2 = 1$.

If we let the lamplighter stand at 0 with all the lamps turned off, this configuration is called the *empty lampstand* and is denoted $e$. See Figure 2.

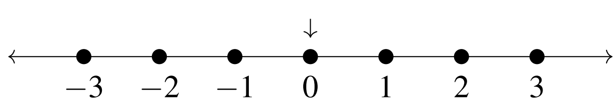


Figure 2: *The empty lampstand e*

**Example 1.** Consider the lampstand $l_1$ in Figure 3.

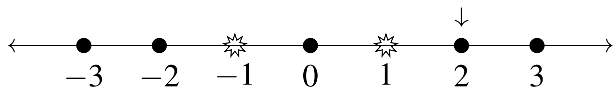


Figure 3: *The lampstand* $l_1$

Starting with the empty lampstand $e$, we can apply a composition of functions $\tau$, $\tau^{-1}$, $\sigma$ and $I$ to achieve $l_1$. For instance the composition $\tau\sigma\tau\tau\sigma\tau^{-1}$ (or $\tau \circ \sigma \circ \tau \circ \tau \circ \sigma \circ \tau^{-1}$) applied to $e$ yields the lampstand configuration $l_1$. In keeping with standard function notation, the order of the composition is such that $\tau^{-1}$ is applied to $e$ first and so on, reading from right to left. Figure 4 shows the details of the transformation from $e$ to $l_1$.

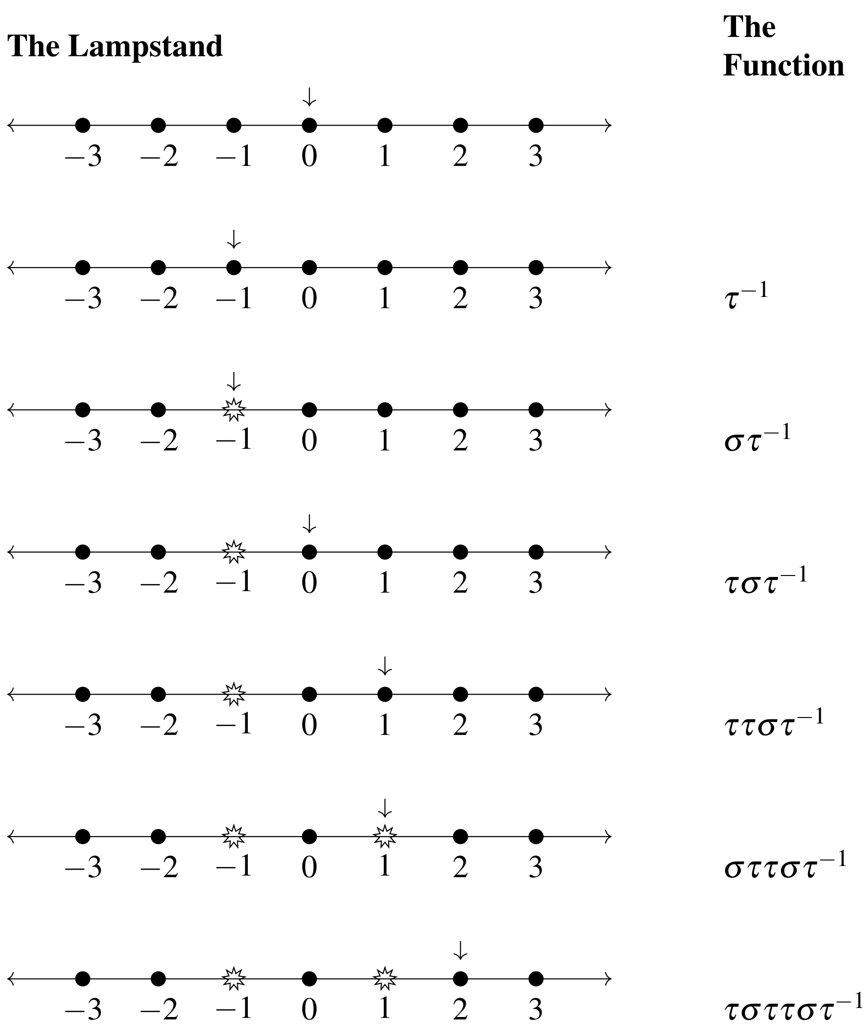


Figure 4: *A sequence of lampstands from the empty lampstand to* $l_1$ $\diamond$

To get the same lampstand $l_1$ (Figures 3 and 4), we could easily have applied a different function composition to $e$, for instance

$$\tau I \tau\tau I \sigma\tau^{-1}\tau^{-1}\sigma\tau.$$

For that matter, pick any $l \in \mathscr{L}$ as input. These two different-looking functions always have the same output.

$$\tau\sigma\tau\tau\sigma\tau^{-1}(l) = \tau I\tau\tau I\sigma\tau^{-1}\tau^{-1}\sigma\tau(l).$$

It doesn't matter that there are different function compositions representing the same lampstand, since two functions are defined to be the same function as long as the domains are the same and the outputs are the same. However, some function compositions are clearly "shorter" than others. Here "shorter" refers to the number of tasks in the function composition. This begs the question, is there a "shortest" function composition for a given lampstand configuration? You will explore this in the Exercise 4 below.

## 2 Assignment #2

**This assignment is due on XX/XX/20XX at 10 am - at the beginning of our class period. You may submit it electronically as a pdf document or as a hard copy. Assignments late by 1 day will be penalized by 25%, 2 days late 50%, 3 days late 75% and any later they will no longer be accepted. You must submit your OWN work, you may not submit a group report.**

**Exercise 1.** Check that the lampstand $l_2$ shown in Figure 5 can be arrived at by starting with the empty lampstand and applying this composition of functions - show/draw all steps: [20 points]

$$\sigma\tau\sigma\tau^{-1}\sigma\tau^{-1}\tau^{-1}\sigma\tau\sigma.$$

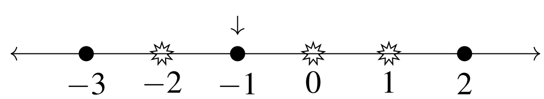


Figure 5: *The lampstand $l_2$*

**Exercise 2.** Prove that $\tau$ is bijective. Please be sure to write out your reasoning and show all of your steps. [30 points]

**Exercise 3.** Draw your own lampstand configuration and write down its function composition (consisting of $\tau$, $\tau^{-1}$ and $\sigma$) and **share it** with one of your team members. You must also include: **the drawing of the lampstand** and **its function composition** here with this assignment in order to receive credit for this part. [10 points]

**Exercise 4.** a. Consider the lampstand shared with you by your fellow team member. Write down **two more** distinct function compositions (consisting of $\tau$, $\tau^{-1}$ and $\sigma$) besides the one shared with you to represent your fellow team member's lampstand. [10 points]

b. Of your three different function compositions (from part a.) representing the lampstand shared with you, which is the "shortest"? How many function compositions of $\tau$, $\tau^{-1}$ and $\sigma$ does it have? [5 points]

c. What is the minimum number of function compositions necessary to express the lampstand shared with you? [5 points]

d. Describe a general method for finding the shortest function composition to represent *any* lampstand $l$ where the lamplighter stands to the left of 0, with finitely many positive and negative lamps lit. [10 points]

e. How would you modify your method if the starting position of the lamplighter was to the right of 0? [10 points]

*Please be sure this writing is your own - do NOT borrow from a friend. I want to hear your own voice, not read a copy and paste of some other source!!!*