New York City College of Technology

Head Start to Calculus II

A Preparatory Workshop for MAT 1575 Calculus II

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In Memory of our beloved friend and colleague

Professor Janet Liou-Mark

Whose devotion and contribution to the education of City Tech students will be remembered

This workbook is created to provide students with a review and an introduction to Calculus II before the actual Calculus II course. Studies have shown that students who attend a preparatory workshop for the course tend to perform better in the course. The Calculus II workshop bears no college credits nor contributes towards graduation requirement. It may not be used to substitute for nor exempt from the Calculus II requirement.

The Calculus II workshop meets four days, three hours a day, for a total of 12 hours during the week before the start of the semester. The workshop is facilitated by instructors and/or peer leaders.

Section 1: Review of Derivatives

One would often see the following terminology when referring to derivatives.

Terminology: Derivative, differentiation, $y', f', \frac{dy}{dx}, \frac{d}{dx}[f(x)]$, or higher order derivatives: $y'',\ y''',\ y^{(4)}, \dots f'', f''', f^{(4)} \dots, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3} \dots$.

General Differentiation Formulas

Derivative of a Constant Function	$\frac{d}{dx}(\boldsymbol{c}) = 0$
The Power Rule for Derivative	$\frac{d}{dx}(x^n) = nx^{n-1}$ where *n* is any real number
The Constant Multiple Rule for Derivative	$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x)$
The Sum Rule for Derivative	$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$
The Difference Rule for Derivative	$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$

The Product Rule	$\frac{d}{dx}[f(x)g(x)] = \frac{d}{dx}[f(x)] \cdot g(x) + \frac{d}{dx}[g(x)] \cdot f(x)$	Or in prime notation $(fg)' = f'g + g'f$
The Quotient Rule	$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{\frac{d}{dx}[f(x)]\cdot g(x) - \frac{d}{dx}[g(x)]\cdot f(x)}{[g(x)]^2}$	Or in prime notation $\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}$
The Chain Rule	$\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$	Or, if y is a function of u, and u is a function of x, then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

Exercise 1: Use the indicated rule to find y'.

a) The Product Rule: $y = x^4(2x + 3)$

b) The Quotient Rule: $y = \frac{9x^2}{3x^2 - 2x}$

c) The Chain Rule: $y = \sqrt{3x^2 + 1}$

d) Combination Rules: $y = \frac{\sqrt{x^2+1}}{(5x+1)^3}$

Derivatives of Trigonometric Functions	
Sine	$\frac{d}{dx}\sin(x) = \cos(x)$
Cosine	$\frac{d}{dx}\cos(x) = -\sin(x)$
Tangent	$\frac{d}{dx}\tan(x) = \sec^2(x)$
Cotangent	$\frac{d}{dx}\cot(x) = -\csc^2(x)$
Secant	$\frac{d}{dx}\sec(x) = \sec(\mathrm{x})\tan(x)$
Cosecant	$\frac{d}{dx}\csc(x) = -\csc(\mathrm{x})\cot(x)$

Derivatives of Inverse Trigonometric Functions	
Inverse Sine	$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$
Inverse Cosine	$\frac{d}{dx}\cos^{-1}(x) = \frac{-1}{\sqrt{1-x^2}}$
Inverse Tangent	$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{\sqrt{1+x^2}}$
Inverse Cotangent	$\frac{d}{dx}\cot^{-1}(x) = \frac{-1}{\sqrt{1+x^2}}$
Inverse Secant	$\frac{d}{dx}\sec^{-1}(x) = \frac{1}{\lvert x\rvert\sqrt{x^2-1}}$
Inverse Cosecant	$\frac{d}{dx}\csc^{-1}(x) = \frac{-1}{\lvert x\rvert\sqrt{x^2-1}}$

Derivatives of Exponential and Natural Logarithmic Functions	
Exponential e	$\frac{d}{dx}e^x = e^x$
Natural Logarithm	$\frac{d}{dx}\ln x = \frac{1}{x}$

Exercise 2: Find the derivative of each function below.

a) $k(x) = \csc x \cdot \cot x$

b) $h(x) = \frac{\cos(2x)}{\sin(3x)+1}$

c) $r(x) = e^x \cos(2x^2)$

d) $f(x) = \frac{\ln x}{x^3}$

e) $t(x) = \cot^{-1}(x^3)$

f) $g(x) = x^2 \sin^{-1} x$

g) A very long Chain Rule: $y = \sin^5\left(\tan\left((e^x + 1)^4\right)\right)$

SECTION 1 SUPPLEMENTARY EXERCISES

1. Find the derivative using the Power Rule. Rewrite each term as an exponent if necessary.

   a) $f(x) = 5x^{-3} + 3x^{-6} - 2$

   b) $m(x) = x^{\frac{-3}{2}} + 3x^{\frac{1}{6}}$

   c) $y = 6\sqrt{x} - \sqrt[3]{x}$

   d) $y = \frac{2}{\sqrt[3]{x}} + 9x$

   e) $s(t) = t^2 + \frac{5}{t^2}$

   f) $y = \frac{x^3-4x^2+8}{x^2}$ (Do not use quotient rule!)

   g) $f(x) = \frac{5x^2-2x+1}{x}$ (Do not use quotient rule!)

2. Find the derivative using the Product or Quotient Rule.

   a) $h(t) = (4t+3)(t-7)$

   b) $y = 3x\sqrt{x+5}$

   c) $p(x) = \frac{x+5}{x^2-9}$

   d) $y = \frac{x^2}{\sqrt{x+8}}$

3) Find the derivative.

   a) $v(x) = (2-4x)^{100}$

   b) $v(x) = -x^3(2-4x)^{100}$

   c) $y = \sqrt{x^2+3x+4}$

   d) $v(x) = \tan\left(\sqrt{x^3+2}\right)$

   e) $n(x) = 5\cos^3(x) - \sin(2x)$

   f) $g(x) = 2x^2\sec^2(8x)$

   g) $f(x) = \frac{\sin(x)\sec(5x)}{3x^2}$

   h) $y = \frac{\cos(\mathrm{x})}{2\sin(-3x)}$

   i) $y = \sqrt{\sin^2(x)+5}$

   j) $f(x) = x^3\sqrt{\tan x + 5}$

   k) $t(x) = \csc^{-1}(2x^4)$

   l) $g(x) = x^3\tan^{-1}x$

   m) $r(x) = e^x\cot(x^2)$

   n) $f(x) = \frac{\ln x}{x}$

Section 2: Antiderivatives

Let F be an antiderivative of f. The indefinite integral of $f(x)$ with respect to x, is defined by:

$$\int f(x)\,dx = F(x) + C \qquad C \text{ is some arbitrary constant}$$

One would often see the following terminology when referring to antiderivative.

Terminology: Antiderivative, integral, integration

General Integration Formulas	
Integral of a Constant Function	$\int f(x)dx = F(x) + C$
The Power Rule for Integral	$\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$
The Constant Multiple Rule for Integral	$\int cf(x)dx = c\int f(x)dx$
The Sum Rule for Integral	$\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx$
The Difference Rule for Integral	$\int (f(x) - g(x))dx = \int f(x)dx - \int g(x)dx$

Trigonometric Integrals
$\int \sin x\,dx = -\cos x + C$
$\int \cos x\,dx = \sin x + C$
$\int \sec^2 x\,dx = \tan x + C$
$\int \csc^2 x\,dx = -\cot x + C$
$\int \sec x \tan x\,dx = \sec x + C$
$\int \csc x \cot x\,dx = -\csc x + C$

Integration Resulting in Inverse Trigonometric Functions
$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{\lvert a\rvert} + C$
$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$
$\int \frac{1}{x\sqrt{x^2 - a^2}} dx = \frac{1}{\lvert a\rvert}\sec^{-1}\frac{\lvert x\rvert}{a} + C$

Exponential and Natural Logarithmic Integrals	
Exponential e	$\int e^x dx = e^x + C$
Natural Logarithm	$\int \frac{1}{x} dx = \ln\lvert x\rvert + C$

Exercise 1. Simplify and rewrite each term as x^n with exponent in the numerator. Find the integral.

Function	Rewriting the exponents (except for x^{-1})	Integral
$\int \left(\frac{1}{x^2} + \frac{1}{x}\right) dx$	$\int x^{-2} + \frac{1}{x}\, dx$	$\frac{x^{-2+1}}{-2+1} + \ln x + C = -\frac{1}{x} + \ln x + C$
$\int \left(\frac{1}{\sqrt{x}} - \sqrt{x}\right) dx$		
$\int \left(\frac{1}{\sqrt[3]{x}} + \left(\sqrt{x}\right)^3\right) dx$		
$\int \frac{3x^2 + 6 + 2x^{-3}}{x^2} dx$		
$\int \frac{x^2 + 6x - 8}{\sqrt{x}} dx$		

Example: Indefinite Integral of a Sum

$$\int (\cos(x) + 2x^5)\,dx = \int \cos(x)\ dx + \int 2x^5\,dx$$
$$= \sin(x) + \frac{2x^6}{6} + C$$
$$= \sin(x) + \frac{x^6}{3} + C$$

Example: Indefinite Integrals of a Difference.

$$\int (11x^3 - 3\csc^2 x)\,dx = \int 11x^3\,dx - \int 3\csc^2 x\,dx$$
$$= \frac{11x^4}{4} - 3(-\cot x) + C$$
$$= \frac{11x^4}{4} + 3\cot x + C$$

Exercise 2: Find the antiderivative of the following functions.

a) $\int (\csc^2 x + 2x^3)\,dx$

b) $\int (2\sec x \tan x - 1)\,dx$

c) $\int (6x - 7\csc x \cot x)dx$

d) $\int \frac{5}{9+x^2}dx$

e) $\int \frac{1}{2\sqrt{4-x^2}}dx$

f) $\int \frac{1}{x\sqrt{x^2-4}}dx$

g) $\int 3e^x + x^e\ dx$

h) $\int \frac{x}{6x^2}dx$

Section 3: Definite Integrals

Let F be an antiderivative of f. For any f function defined on $[a, b]$, the definite integral of f from a to b is defined by:

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

Example: Evaluate the definite integral

$$\int_1^2 x^2 - x\,dx \;=\; \frac{x^3}{3} - \frac{x^2}{2}\Big|_1^2$$

$$= \left(\frac{(2)^3}{3} - \frac{(2)^2}{2}\right) - \left(\frac{(1)^3}{3} - \frac{(1)^2}{2}\right)$$

$$= \left(\frac{8}{3} - 2\right) - \left(\frac{1}{3} - \frac{1}{2}\right) \;=\; \frac{7}{3} - 2 + \frac{1}{2} = \frac{5}{6}$$

Exercise 1: Evaluate the definite integral

1. $\int_0^3 (x^4 + 3x)\,dx$

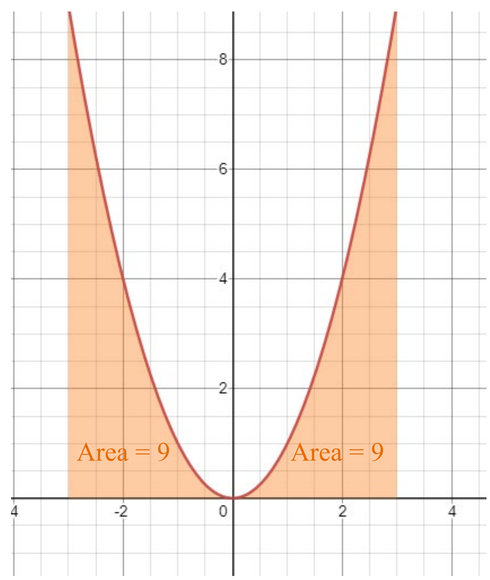
2. $\int_1^8 \left(\sqrt[3]{x}\right)^2\,dx$

c) $\int_0^2 e^{x} dx$

d) $\int_0^{2\pi} \left(\frac{x}{3} - \cos(x)\right) dx$

Exercise 2: Compare the following definite integrals:

a) $\int_{-3}^{3} x^{2}\, dx$



b) $\int_{3}^{-3} x^{2}\, dx$

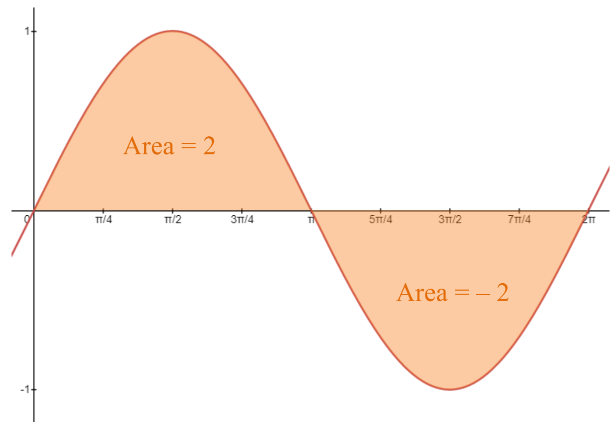
c) $\int_{-3}^{0} x^{2}\, dx + \int_0^3 x^{2}\, dx$

The definite integral represents the area between the function and the x-axis. The area above the x-axis is positive, the area below the x-axis is negative.

Properties of definite integrals:

Properties Similar to the General Properties of Integrals	Properties Specific to Definite Integrals
$\int_a^b cf(x)\,dx = c\int_a^b f(x)dx$	$\int_a^a f(x)dx = 0$
$\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)dx + \int_a^b g(x)dx$	$\int_b^a f(x)dx = -\int_a^b f(x)dx$
$\int_a^b (f(x)-g(x))\,dx = \int_a^b f(x)dx - \int_a^b g(x)dx$	$\int_a^b f(x)\,dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

Exercise 3: Compare the following definite integrals:



a) $\int_0^{2\pi} \sin x \; dx$

b) $\int_{2\pi}^{0} \sin x \; dx$

c) $\int_0^{\pi} \sin x \; dx + \int_{\pi}^{2\pi} \sin x \; dx$

SECTION 2 SUPPLEMENTARY EXERCISES

Find the integral

1. $\int 45x^4\, dx$
2. $\int \frac{2x^6}{5}\, dx$
3. $\int 5\sqrt{x}\, dx$
4. $\int (x^5 + \sqrt{x}\,)\, dx$
5. $\int (4x^{-2} - \frac{1}{\sqrt{x}})\, dx$

SECTION 3 SUPPLEMENTARY EXERCISES

Evaluate the definite integral

1. $\int_{-3}^{3} \left(3 + \sqrt{x^3}\right) dx$
2. $\int_{-4}^{-1} \left(3x^{-2} - \frac{1}{x}\right) dx$

Section 4: Area Between Curves

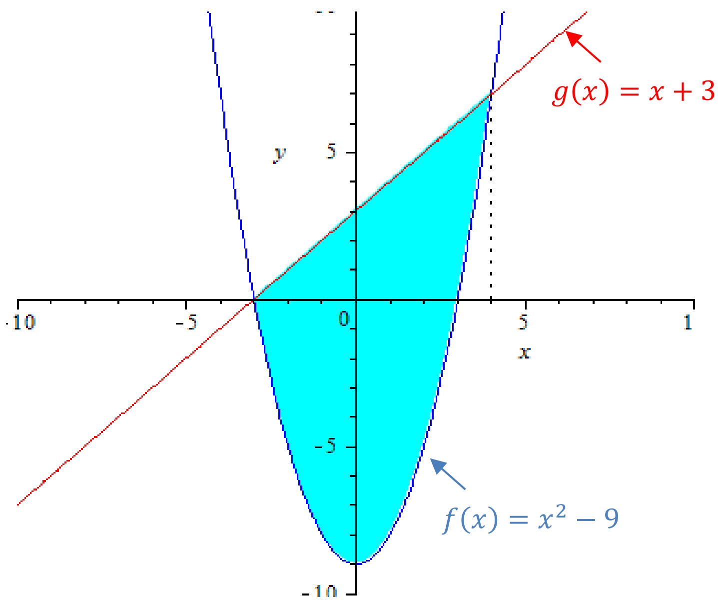
The area between curves is determined by integrating the function (or the difference of two functions) on a given interval $[a, b]$, which corresponds to the limits of integration.

$$\int_a^b |f(x) - g(x)|\, dx$$

It is always helpful to sketch a graph before setting up the integral.

Area of a Region between Two Curves: When one function is always greater than or equal to the other function on the interval $[\boldsymbol{a}, \boldsymbol{b}]$

Example: Find the area between the two curves $f(x) = x^2 - 9$ and $g(x) = x + 3$.



Step 1: Sketch a graph.

Step 2: Find where the two curves intersect. That gives the interval or the limits of integration. Set the two functions equal to each other and solve for the variable.

$$f(x) = g(x)$$
$$x^2 - 9 = x + 3$$
$$x^2 - x - 12 = 0$$
$$(x + 3)(x - 4) = 0$$
$$x = -3, x = 4$$

The interval is $[-3, 4]$, corresponding to the limits of integration $a = -3$ and $b = 4$.

Step 3: Set up the integral. Since $g(x)$ is greater than $f(x)$ on the interval $[-3,4]$, we set up:

$$\int_a^b (g(x)-f(x))\,dx = \int_{-3}^{4} \big((x+3)-(x^2-9)\big)\;dx$$

Step 4: Evaluate.

$$\int_{-3}^{4} \big((x+3)-(x^2-9)\big)\;dx = \int_{-3}^{4} (-x^2+x+12)\,dx = \left(-\frac{x^3}{3}+\frac{x^2}{2}+12x\right)\Big|_{-3}^{4}$$

$$= \left(-\frac{(4)^3}{3}+\frac{(4)^2}{2}+12(4)\right)-\left(-\frac{(-3)^3}{3}+\frac{(-3)^2}{2}+12(-3)\right)$$

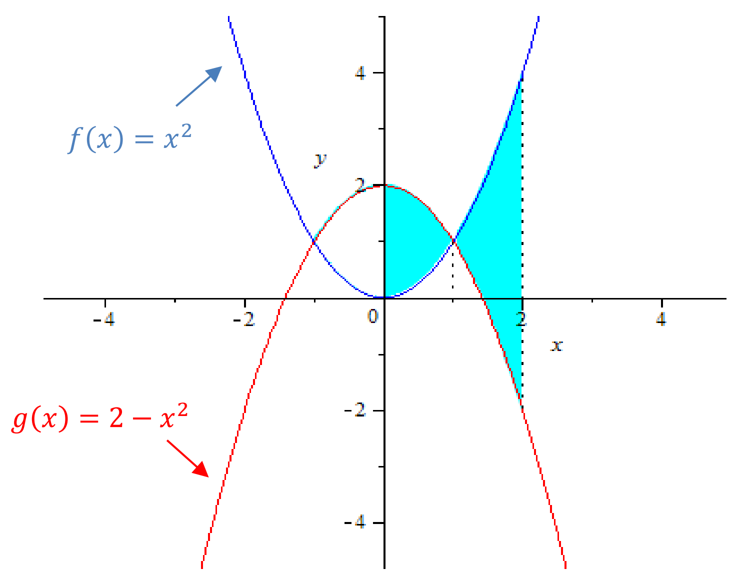
$$\left(-\frac{64}{3}+\frac{16}{2}+48\right)-\left(\frac{27}{3}+\frac{9}{2}-36\right)=\frac{343}{6}$$

Thus, the area between $f(x)$ and $g(x)$ is $\frac{343}{6}$.

Areas of a Region Bounded by Functions That Cross

It two curves cross creating multiple regions, we would need to determine where they intersect and set up separate integral for each region.

Example: Find the area between $f(x)=x^2$ and $g(x)=2-x^2$ on the interval $[0,2]$.



Step 1: Sketch a graph.

Step 2: Find where the two curves intersect. Set two functions equal to each other and solve for x.

$$x^2 = 2 - x^2$$

$$x = -1, x = 1$$

Note $g(x) \geq f(x)$ on the interval $[0,1]$ and $f(x) \geq g(x)$ on the interval $[1,2]$. We need to set up separate integrals and use the absolute value to get a positive integral value.

$$\int_0^1 |f(x) - g(x)| dx + \int_1^2 |f(x) - g(x)|\, dx$$

$$\int_0^1 |(x^2) - (2 - x^2)|\, dx + \int_1^2 |(x^2) - (2 - x^2)|\, dx = \int_0^1 |2x^2 - 2|\, dx + \int_1^2 |2x^2 - 2|\, dx$$

$$= \left|\frac{2x^3}{3} - 2x\right|\Big|_0^1 + \left|\frac{2x^3}{3} - 2x\right|\Big|_1^2$$

$$= \left|-\frac{4}{3}\right| + \left|\frac{8}{3}\right| = \frac{4}{3} + \frac{8}{3} = 4$$

Therefore, the area between $f(x)$ and $g(x)$ where $0 \leq x \leq 2$ is 4.

Note that, the integral on the interval $[0,1]$ gives a negative area. Without the absolute value, we would have obtained an incorrect answer.

Exercise 1: Find the area between the curves on the given interval. Sketch a graph.

a) $f(x) = x^3, g(x) = x$ on the interval $[0, 1]$.

b) $f(x) = x^2 - 3$ and $g(x) = 5 - x^2$ on the interval $[-2, 2]$.

c) $f(x) = \sin(x)$ and $g(x) = x^2$ on the interval $[0, 1]$.

d) Find the area between the curves determined by the intersection of $f(x) = x^2$ and $g(x) = \sqrt{x}$.

SECTION 4 SUPPLEMENTARY EXERCISES

1. Find the area between the curves determined by the intersection

   a) $f(x) = 7 - x^2,\ g(x) = x^2 - 1$

   b) $f(x) = x, g(x) = x^2 - 16$

Section 5: Integration by Substitution

The variable u is commonly in the method of integration by substitution and therefore leads to the general expression "u – substitution" for the "innermost" term of a composition of a function.

Integration by the method of substitution consists of the following general steps.

- Let u be a function of x. It represents the "innermost" or the "inside" function in the integral.
- Compute $du = \frac{du}{dx}$ with respect to x.
- Substitute all terms of the original integrand with the expressions involving u and du.
- Evaluate the integral involving the u and du expressions.
- Replace all substitutions of u with the original variable expression.

Integration by Substitution

Example: Integrate using the method of substitution (or u – substitution)

$$\int (x^5 + 100)^{50}(5x^4)\, dx$$

Check that $\frac{d}{dx}(x^5 + 100) = 5x^4$ is a part of the integrand.

Step 1: Let $u = x^5 + 100$ and $du = 5x^4 dx$ by the Power Rule of differentiation.

Step 2: Substitute expression using u and du.

$$\int (x^5 + 100)^{50}(5x^4)\, dx = \int u^{50} du$$

Step 3: Evaluate:

$$\int u^{50} du = \frac{u^{51}}{51} + c$$

Step 4: Replacing $u = x^5 + 100$ and we have the answer

$$\int (x^5 + 100)^{50}(5x^4)\, dx = \frac{(x^5 + 100)^{51}}{51} + C$$

Example: Integrate using the method of substitution (or u – substitution)

$$\int (2\sin(x) - 7)^{15} \cos(x)\, dx$$

Check that $\frac{d}{dx}(2\sin(x) - 7) = 2\cos(x)$ is a part of the integrand.

Step 1: Let $u = 2\sin(x) - 7$ and $du = 2\cos(x)dx$.

Step 2: In the integrand, we have $\cos(x)dx$ but not $2\cos(x)dx$. In order to have $2\cos(x)dx$, we multiply the interior of the integral by 2 and divide on the exterior of the integral by 2. Substitute expression using u and du.

$$\int (2\sin(x) - 7)^{15} \cos(x)\, dx = \frac{1}{2}\int (2\sin(x) - 7)^{15}\, 2\cos(x)\, dx = \frac{1}{2}\int u^{15} du$$

Step 3: Evaluate:

$$\frac{1}{2}\int u^{15} du = \frac{1}{2}\left(\frac{u^{16}}{16}\right) + C = \frac{u^{16}}{32} + C$$

Step 4: Replacing $u = 2\sin(x) - 7$, we have the answer

$$\int (2\sin(x) - 7)^{15} \cos(x)\, dx = \frac{(2\sin x - 7)^{16}}{32} + C$$

Exercise 1: Integrate using the method of substitution (u –substitution):

a) $\int x^4\sqrt{x^5 - 9}\, dx$

b) $\int e^{-3x}\ dx$

c) $\int \frac{sin\sqrt{x}}{\sqrt{x}}\ dx$

d) $\int \tan x\ \ dx$ (Hint: rewrite $\tan x = \frac{\sin x}{\cos x}$)

Exercise 2: Compare and integrate the following integrals. What method would you use to integrate each?

a) $\int \frac{x}{\sqrt{1+x^2}}dx$

b) $\int \frac{x}{1+x^2}dx$

c) $\int \frac{1}{1+x^2}dx$

Integration by Variable Substitution of Definite Integrals

Like the method of substitution for indefinite integrals, definite integral is very similar. The only difference is that we need to adjust the limits of integration accordingly. There are two methods in handling the limits of integration.

The first method is to make no change in the limits of integration. We would integrate completely and substitute back $u(x)$, then evaluate definite integral using the original limits of integration.

The second method is to change the limits of integration to correspond to $u(x)$. See below. The definite integral is evaluated using the new limits of integration.

$$\int_a^b f\big(u(x)\big)u'(x)dx = \int_{u(a)}^{u(b)} f(u)du$$

Example: Integrate using the method of substitution

$$\int_1^3 3x^2\sqrt{x^3+3}\ dx$$

Notice: $\frac{d}{dx}\left(x^3+3\right) = 3x^2$, which is a part of the integrand.

Let $\boldsymbol{u = x^3+3}$ and $\boldsymbol{du = 3x^2dx}$, we make our substitution

$$\int \boldsymbol{3x^2\sqrt{x^3+3}\ dx} = \int \sqrt{\boldsymbol{u}}\ \boldsymbol{du}$$

Method 1: Keep the original limits of integration

- Integrate without the limits.
- Substitute back $u(x)$.

$$\int 3x^2\sqrt{x^3+3}\ dx = \int \sqrt{u}\ du = \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C = \frac{2\left(x^3+3\right)^{\frac{3}{2}}}{3} + C$$

- Evaluate the definite integral using the original limits.

$$\int_1^3 3x^2\sqrt{x^3+3}\ dx = \left.\frac{2\left(x^3+3\right)^{\frac{3}{2}}}{3}\right|_1^3 = \frac{2\left(3^3+3\right)^{\frac{3}{2}}}{3} - \frac{2\left(1^3+3\right)^{\frac{3}{2}}}{3} = 20\sqrt{30} - \frac{16}{3}$$

Method 2: Change the limits of integration to $u(a)$ and $u(b)$.

$$\int_1^3 3x^2\sqrt{x^3+3}\ dx = \int_{u(1)}^{u(3)} \sqrt{u}\, du$$

Since $u = x^3+3$, substitute $a = 1$ and $b = 3$, hence, $u(1) = 4$ and $u(3) = 30$.

$$\int_1^3 3x^2\sqrt{x^3+3}\ dx = \int_4^{30} \sqrt{u}\, du$$

Evaluate the adjusted definite integral:

$$\int_4^{30} \sqrt{u}\, du = \left.\frac{2x^{\frac{3}{2}}}{3}\right|_4^{30} = \frac{2(30)^{\frac{3}{2}}}{3} - \frac{2(4)^{\frac{3}{2}}}{3} = 20\sqrt{30} - \frac{16}{3}$$

Exercise 3: Evaluate the following definite Integrals using substitution (u – substitution):

a) $\int_0^1 x\sqrt{x^2+2}\ dx$

b) $\int_{-1}^1 \frac{x}{(x^2+1)^2}\ dx$

c) $\int_0^{\frac{\pi}{2}} \sec^2 5x\, dx$

SECTION 5 SUPPLEMENTARY EXERCISES

1. Find the Integral

   a) $\int x^3(x^4-3)^{-\frac{2}{3}}\,dx$

   b) $\int \frac{x}{\sqrt{1-x^2}}\,dx$

   c) $\int \frac{1}{\sqrt{1-x^2}}\,dx$

   d) $\int \frac{2\sin(x)}{\sqrt{\cos(x)}}\,dx$

   e) $\int \frac{1}{x\ln x}\,dx$

   f) $\int \frac{e^x}{1+e^x}\,dx$

   g) $\int \frac{\sin x}{1+\cos x}\,dx$

2. Evaluate the definite Integral

   a) $\int_0^4 x^2\cos x^3\,dx$

   b) $\int_1^5 \frac{1}{2x-1}\,dx$

   c) $\int_0^1 x^3 e^{x^4}\,dx$

Section 6: Integration by Parts

Many integrals cannot be evaluated using basic formulas or integration by substitution. We introduce another method to help evaluate integrals, the method of integration by parts.

Integration by Parts:

$$\int u\,dv = uv - \int v\,du$$

Example: The integral below has no simplification or substitution, but we can integrate by parts

$$\int x \sin(x)\,dx$$

First we need to determine the four "objects" (u, v, du, dv). The givens are u and dv. We need to carefully choose which part of our function will be u (something to differentiate to obtain du) and which will be dv (something to integrate to obtain v). Poor choice of u and dv will most likely lead to more complications. Let us fit the question to our formula using color-codes:

$$\int \boldsymbol{x}\ \mathbf{sin}\,\boldsymbol{x}\,\boldsymbol{dx} = \boldsymbol{uv} - \int \boldsymbol{v}\,\boldsymbol{du}$$

$\boldsymbol{u = x}$ $\qquad$ $\boldsymbol{dv = \sin(x)dx}$

$\boldsymbol{du = 1dx}$ $\qquad$ $\boldsymbol{v = -\cos(x)}$

$$\int \boldsymbol{x}\ \mathbf{sin}\,\boldsymbol{x}\,\boldsymbol{dx} = \boldsymbol{x(-\cos x)} - \int \boldsymbol{(-\cos(x))1dx}$$

Simplify and complete the integration:

$$-x\cos(x) + \int \cos(x)\,dx \;=\; -x\cos(x) + \sin(x) + C$$

Exercise 1: Integrate the following integrals by Parts

a) $\int x\cos(x)\,dx$

$u =$ $dv =$

$du =$ $v =$

b) $\int x\ln(x)\,dx$ How do you determine which part to be u and which part dv?

$u =$ $dv =$

$du =$ $v =$

c) $\int x\,sec^2(x)dx$

$u =$ $dv =$

$du =$ $v =$

Exercise 2: Which method would you use to integrate the integrals below? Integration by Substitution or Integration by Parts? Evaluate the integrals.

a) $\int x\, e^{x} dx$

b) $\int x\, e^{x^2} dx$

Exercise 3: Which method would you use to integrate the integrals below? Integration by Substitution or Integration by Parts? Evaluate the integrals.

a) $\int \ln(x)\, dx$

b) $\int \frac{\ln(x)}{x} dx$

SECTION 6 SUPPLEMENTARY EXERCISES

Evaluate the following definite Integrals using integration by parts:

a) $\int x \sin(5x)\, dx$

b) $\int x^2\, e^x\, dx$

c) $\int x \sec^2(x) dx$

Section 7: Trigonometric Integrals

Helpful Trigonometric Identities

The Reciprocal Relations:

$$\sin x = \frac{1}{\csc x} \qquad \cos x = \frac{1}{\sec x} \qquad \tan x = \frac{1}{\cot x}$$

The Quotient Relations:

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x}$$

Pythagorean Relations:

$$\sin^2 x + \cos^2 x = 1$$
$$\tan^2 x + 1 = \sec^2 x$$
$$\cot^2 x + 1 = \csc^2 x$$

Double Angle Identities:

$$\cos 2x = 2\cos^2 x - 1$$
$$\cos 2x = 1 - 2\sin^2 x$$

The tables below provide strategies for simplifying and integrating trigonometric integrals. Note that m is an integer not equal to zero. Also note that there may be multiple ways to integrate a problem.

Trigonometric Integral	u-substitution
$\int \sin^m x \cos x \, dx$	$u = \sin x,\ du = \cos x \, dx$
$\int \cos^m x \sin x \, dx$	$u = \cos x,\ du = -\sin x \, dx$
$\int \tan^m x \sec^2 x \, dx$	$u = \tan x,\ du = \sec^2 x \, dx$
$\int \cot^m x \csc^2 x \, dx$	$u = \cot x,\ du = -\csc^2 x \, dx$
$\int \sec^m x \tan x \, dx = \int \sec^{m-1} x \sec x \tan x \, dx$	$u = \sec x,\ du = \sec x \tan x \, dx$
$\int \csc^m x \cot x \, dx = \int \csc^{m-1} x \csc x \cot x \, dx$	$u = \csc x,\ du = -\csc x \cot x \, dx$

Use the double angle identities to rewrite the integrals:

Trigonometric Integral	Double Angle Identity
$\int \sin^2 x\, dx$	$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos(2x)$
$\int \cos^2 x\, dx$	$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos(2x)$

Use the Pythagorean trigonometric identities, then the u-substitution:

Trigonometric Integral	Pythagorean Trig Identity	
$\int \sin^3 x\, dx$	$\sin^2 x = 1 - \cos^2 x$	$\int \sin^3 x\, dx = \int \sin x\,(1 - \cos^2 x)\, dx$
$\int \sin^3 x\, \cos^m x\, dx$	$\sin^2 x = 1 - \cos^2 x$	$\int \sin^3 x \cos^m x\, dx = \int \sin x\,(1 - \cos^2 x) \cos^m x\, dx$
$\int \cos^3 x\, dx$	$\cos^2 x = 1 - \sin^2 x$	$\int \cos^3 x\, dx = \int \cos x\,(1 - \sin^2 x)\, dx$
$\int \cos^3 x \sin^m x\, dx$	$\cos^2 x = 1 - \sin^2 x$	$\int \cos^3 x \sin^m x\, d = \int \cos x\,(1 - \sin^2 x) \sin^m x\, dx$
$\int \tan^3 x\, dx$	$\tan^2 x = \sec^2 x - 1$	$\int \tan^3 x\, dx = \int \tan x\,(\sec^2 x - 1)\, dx$
$\int \tan^3 x\, \sec^m x\, dx$	$\tan^2 x = \sec^2 x - 1$	$\int \tan^3 x \sec^m x\, dx = \int \tan x\,(\sec^2 x - 1) \sec^m x\, dx$

Exercise 1: Integrate each trigonometric integral below. Clearly indicate u and du in the u-substitutions

a) $\int \sin^3(x) \cos(x)\, dx$

b) $\int \sin^3(x) \cos^2(x)\, dx$

Exercise 2: Integrate $\int \sin^2 x \, dx$

Exercise 3: Integrate $\int \tan x \sec^2 x \, dx$ using two different u-substitutions.

a) $u = \tan x$, $du = \sec^2 x \, dx$

b) $u = \sec x$, $du = \sec x \tan x \, dx$

c) Show the answers in (a) and (b) are equivalent.

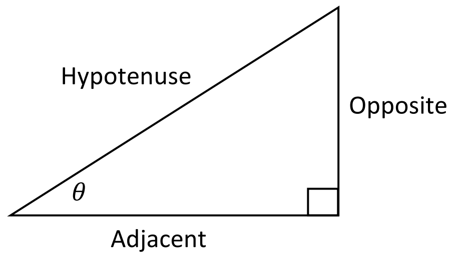
Section 8: Trigonometric Substitution

Defining Trigonometric Functions

$$\sin\theta = \frac{\text{Side Opposite } \theta}{\text{Hypotenuse}} = \frac{\text{O}}{\text{H}}$$

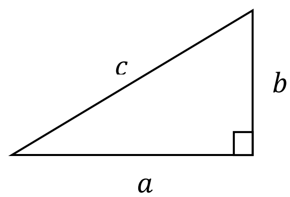
$$\cos\theta = \frac{\text{Side Adjacent to } \theta}{\text{Hypotenuse}} = \frac{\text{A}}{\text{H}}$$

$$\tan\theta = \frac{\text{Side Opposite } \theta}{\text{Side Adjacent to } \theta} = \frac{\text{O}}{\text{A}}$$



The Pythagorean Theorem
For any right triangle with legs a, b and hypotenuse c, the square of the hypotenuse is equal to the sum of squares of the two legs, or

$$a^2 + b^2 = c^2$$

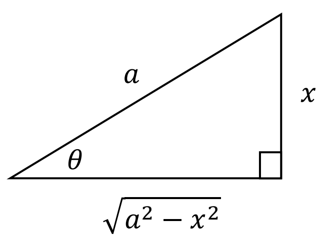


Example 1: Let $x = a\sin\theta$, determine the relationship between the sides of a right triangle.

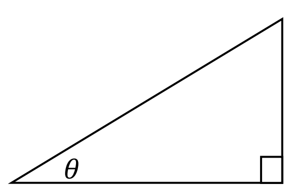
Divide both sides by a, we have

$$\sin\theta = \frac{x}{a} = \frac{\text{Opposite Side}}{\text{Hypotenuse}}$$

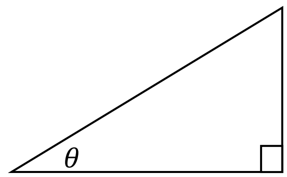
Using the definition of sine and the Pythagorean Theorem, we have the relationship:



Example 2: Let $x = a\tan\theta$, determine the relationship between the sides of a right triangle.



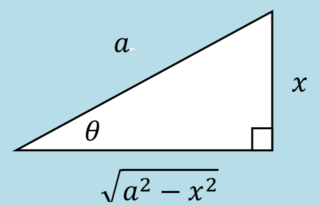
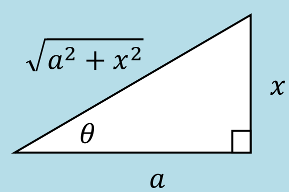
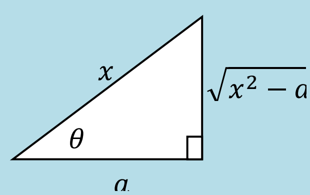
Example 3: Let $x = a \sec \theta$, determine the relationship between the sides of a right triangle.



A trigonometric substitution can often be made to evaluate integrals containing the radical

$$\sqrt{a^2 - x^2},\ \sqrt{a^2 + x^2},\ \text{or}\ \sqrt{x^2 - a^2}\ ,\ \text{for}\ a > 0$$

.

Table of Trigonometric Substitutions			
Expression	Substitution	Identity	Triangle Relationship
$\sqrt{a^2 - x^2}$	$\boldsymbol{x = a \sin \theta}$ $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$	
$\sqrt{a^2 + x^2}$	$\boldsymbol{x = a \tan \theta}$ $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$	
$\sqrt{x^2 - a^2}$	$\boldsymbol{x = a \sec \theta}$ $0 \leq \theta < \frac{\pi}{2}$ or $\pi \leq \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$	

Example: Integrate

$$\int \frac{1}{x^2\sqrt{x^2+4}}\,dx$$

Because the integral has a radical of the form $\sqrt{a^2+x^2}$, we use the following substitution:

$$x = 2\tan\theta \quad \text{and} \quad dx = 2\sec^2\theta\, d\theta$$

Substitute directly into x and dx and change the integral to be in terms of θ. The color shows x and dx.

$$\int \frac{1}{x^2\sqrt{x^2+4}}\,dx = \int \frac{1}{4\tan^2\theta\sqrt{4\tan^2\theta+4}}\,2\sec^2\theta\, d\theta$$

Note, under the radical we can simplify:

$$4\tan^2\theta + 4 = 4\,(\tan^2\theta + 1) = 4\sec^2\theta$$

$$\int \frac{2\sec^2\theta}{4\tan^2\theta\sqrt{4\sec^2\theta}}\,d\theta = \int \frac{2\sec^2\theta}{4\tan^2\theta \cdot 2\sec\theta}\,d\theta = \int \frac{\sec\theta}{4\tan^2\theta}\,d\theta = \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta$$

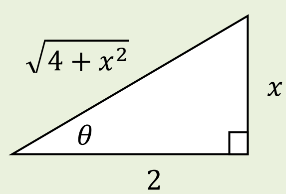
Now we can integrate by substitution. Let $u = \sin\theta$ and $du = \cos\theta d\theta$, then

$$\frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta = \frac{1}{4}\int \frac{du}{u^2} = \frac{1}{4}\int u^{-2}du = \frac{1}{4}(-u^{-1}) + c = -\frac{1}{4u} + C$$

Replacing $u = \sin\theta$, we have the answer in terms of θ.

$$-\frac{1}{4u} + C = -\frac{1}{4\sin\theta} + C$$

But remember, we started with x, not θ. We must convert θ back to x. We look in the triangle relationship for $x = 2\tan\theta$, we have



In this triangle, $\sin\theta = \frac{\text{Opposite}}{\text{Hypotenuse}} = \frac{x}{\sqrt{4+x^2}}$. Substitute $\frac{x}{\sqrt{4+x^2}}$ for $\sin\theta$, we get the final answer:

$$\int \frac{1}{x^2\sqrt{x^2+4}}\,dx = -\frac{1}{4\sin\theta} + C = -\frac{\sqrt{x^2+4}}{4x} + C$$

Exercise 1: Evaluate the following integrals

a) $\int \frac{1}{x^2\sqrt{x^2-9}}\, dx$

b) $\int \frac{1}{x^2\sqrt{16-x^2}}\, dx$

c) $\int \frac{x^2}{\sqrt{x^2+9}}\, dx$

SECTION 7 SUPPLEMENTARY EXERCISES

1. Integrate each trigonometric integral below. Indicate u and du in the u-substitutions

a) $\int \cos^8(x)\sin(x)\,dx$

b) $\int \tan^3(x)\sec^7(x)\,dx$

c) $\int \cos^2 x\,dx$

SECTION 8 SUPPLEMENTARY EXERCISES

1. Use trigonometric substitution to integrate. Evaluate the resulting definite integral.

a) $\int_0^1 \frac{x}{\sqrt{4-x^2}}\,dx$

b) $\int_1^2 \frac{\sqrt{x^2-1}}{x}\,dx$

c) $\int_{\sqrt{2}}^2 \frac{1}{x^3\sqrt{x^2-1}}\,dx$