NEW YORK CITY COLLEGE OF TECHNOLOGY
CITY UNIVERSITY OF NEW YORK

Head Start to Precalculus

A Preparatory Workshop for
MAT 1375 Precalculus

Prepared by

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Head Start to Precalculus

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In Memory of our beloved friend and colleague

Professor Janet Liou-Mark

Whose devotion and contribution to the education of City Tech students will be remembered

This workbook is created to provide students with a review and an introduction to Precalculus before the actual Precalculus course. Studies have shown that students who attend a preparatory workshop for the course tend to perform better in the course. The Precalculus workshop bears no college credits nor contributes towards graduation requirement. It may not be used to substitute for nor exempt from the Precalculus requirement.

The Precalculus workshop meets four days, three hours a day, for a total of 12 hours during the week before the start of the semester. The workshop is facilitated by instructors and/or peer leaders.

Module I

Review of Factoring, Radicals, and Quadratic Equations

Section 1.1 Factoring Review

The Greatest Common Factor

The greatest common factor (GCF) for a polynomial is the largest monomial that divides each term of the polynomial.

Factoring the greatest common factor of a polynomial:

1. **Determine the greatest common factor**
2. **Write the answer in factored form.**

The GCF factoring process is the reverse of the Distributive Property

Multiply

The Distributive Property: $\boldsymbol{a(b+c) = ab + ac}$

Factor/Divide

The GCF factoring: $\boldsymbol{ab + ac = a(b+c)}$

Exercise 1: Factor the greatest common factor and express the answer in factored form. Check by multiplying using the Distributive Property.

a) Factor $5p^3 + 15p^2 - 30p$

b) Factor $9a^3b^4 - 6a^2b^3 + 3ab^2$

c) Factor $5(x+y) - 6x(x+y)$

Factoring by Grouping

Before factoring by grouping, first we factor out GCF from all four terms.

Steps in factoring by grouping (Assume there is no GCF)

1. Group pairs of terms and factor each pair.
2. If there is a common binomial factor, then factor it out.
3. If there is no common binomial factor, then interchange the middle two terms and repeat the process over. If there is still no common binomial, then the polynomial cannot be factored.

Factor $4x + 6y + 2xy + 3y^2$ by grouping

Step 1: Group the pairs of terms	$(\mathbf{4x + 6y}) + (\mathbf{2xy + 3y^2})$
Step 2: Factor the GCF from each pair	$\mathbf{2}(\mathbf{2x + 3y}) + \boldsymbol{y}(\mathbf{2x + 3y})$
Step 3: Factor the common binomial factor	$(\mathbf{2x + 3y})(\mathbf{2 + y})$

Exercise 2: Factor by grouping. Follow the steps in the table

a) Factor by grouping: $56 + 21k + 8h + 3hk$

b) Factor by grouping: $5x^2 + 40x - xy - 8y$

Factoring Trinomials

The FOIL (Front-Outer-Inner-Last) Method: $(a+b)(c+d) = ac + ad + bc + bd$

ac bd

$$(\boldsymbol{a} + \boldsymbol{b})(\boldsymbol{c} + \boldsymbol{d}) = \boldsymbol{ac} + \boldsymbol{ad} + \boldsymbol{bc} + \boldsymbol{bd}$$

bc

ad

Factoring Trinomials with Lead Coefficients of 1

Steps to factoring trinomials with lead coefficient of 1

1. Write the trinomial in descending powers.
2. List the factorizations of the constant (third) term of the trinomial.
3. Pick the factorization where the sum of the factors is the coefficient of the middle term.
4. Check by multiplying the binomials.

Exercise 6: Factor the trinomial with lead coefficient 1 by checking for the correct pair of product and sum

a) $x^2 - 8x + 12$

b) $y^2 - 9y - 36$

c) $x^2 + 14xy + 45y^2$

d) $a^2 + 7ab - 8b^2$

Factoring Trinomials with Lead Coefficients other than 1

<u>Method 1</u>: Trial and error

1. List the factorizations of the third term of the trinomial.
2. Write them as two binomials and determine the correct combination where the sum of the outer product, ad, and the inner product, bc, is equal to the middle term of the trinomial.

$$(\boldsymbol{a} + \boldsymbol{b})(\boldsymbol{c} + \boldsymbol{d}) = \boldsymbol{ac} + \boldsymbol{ad} + \boldsymbol{bc} + \boldsymbol{bd}$$

bc

ad

<u>Method 2</u>: Factoring by grouping

1. Form the product ac.
2. Find a pair of numbers whose product is ac and whose sum is b.
3. Rewrite the polynomial to be factored so that the middle term bx is written as the sum of the two terms whose coefficients are the two numbers found in step
4. Factor by grouping.

Exercise 7: Factor each trinomial below

a) $2h^2 - 5h - 3$

b) $2h^2 - h - 3$

c) $2h^2 - 5h - 12$

d) $3k^2 - 14k - 5$

Factoring Special Products

Difference of Two Squares: $\boldsymbol{a^2 - b^2 = (a + b)(a - b)}$

Exercise 10: Factor each binomial below. Check if it is a difference of two squares.

a) $x^2 - 49$

b) $25 - b^2$

c) $36x^2 - 16y^2$

d) $100t^2 - 49r^2$

e) Factor completely $16x^4 - 81$

f) $x^2 + 36$

What have you noticed about the sum of two squares? ________________________________

Square of a Binomial

$$\boldsymbol{a^2 + 2ab + b^2 = (a+b)^2}$$
$$\boldsymbol{a^2 - 2ab + b^2 = (a-b)^2}$$

Exercise 11: Factor each trinomial below. Check that the first and the third terms are perfect square, and the middle term is $2ab$, then apply the perfect square formula

a) $p^2 + 10t + 25$

b) $9b^2 + 42b + 49$

c) $16a^2 - 40a + 25$

d) $16y^2 - 72y + 81$

Sum of Two Cubes
Difference of Two Cubes

$$\boldsymbol{a^3 + b^3 = (a+b)(a^2 - ab + b^2)}$$
$$\boldsymbol{a^3 - b^3 = (a-b)(a^2 + ab + b^2)}$$

Exercise 12: Factor each trinomial below. Check if they are sum or difference of two cubes

a) Factor $x^3 + 8$

b) Factor $27y^3 - 1$

c) Factor $64x^3 - 27y^3$

Cube of a binomial

$$\boldsymbol{a^3 + 3a^2b + 3ab^2 + b^3 = (a+b)^3}$$
$$\boldsymbol{a^3 - 3a^2b + 3ab^2 - b^3 = (a-b)^3}$$

Exercise 13: Factor each polynomial below. Check if they are cube of a binomial.

a) $x^3 - 3x^2 + 3x - 1$

b) $x^3 + 6x^2 + 12x + 8$

c) $x^3 - 9x^2 + 27x - 27$

Mixed Factoring/Factoring Completely

To factor a polynomial, **first factor the greatest common factor**, then consider the number of terms in the polynomial.

I. **Two terms**: Determine if the binomial is one of the following:
   - Difference of two squares: $a^2 - b^2 = (a+b)(a-b)$
   - Sum of two cubes: $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$
   - Difference of two cubes: $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

II. **Three Terms**: Determine if the trinomial is a perfect square trinomial.
   a) If the trinomial is a perfect square, then
      $a^2 + 2ab + b^2 = (a+b)^2$
      $a^2 - 2ab + b^2 = (a-b)^2$
   b) If the trinomial $ax^2 + bx + c$ is not a perfect square, then
      i) if the leading coefficient $a = 1$, then check the product = c and sum = b to determine the correct combination.
      ii) if the leading coefficient $a > 1$, then use trial and error or the ac-method.

III. **Four terms**: Try to factor by grouping.

Exercise 12: Factor each polynomial **completely**. This may mean factoring GCF and/or factoring in two or more steps.

a) Factor completely $3x^4 - 3x^3 - 36x^2$

b) Factor completely $20a - 5a^3$

c) Factor completely $16a^5b - ab$

d) Factor completely $8x^2 - 24x + 18$

e) Factor completely $16x^3y - 40x^2y^2 + 25xy^3$

f) Factor completely $12x^3 + 11x^2 + 2x$

g) Factor completely $7z^2w^2 - 10zw^2 - 8w^2$

Section 1.2: Radicals and the Complex Numbers

Simplifying Radicals

Exercise 1: Simplify each radical below

a. $\sqrt{9}$

b. $\sqrt{18}$

c. $\sqrt{25}$

d. $\sqrt{75}$

e. $\sqrt{49}$

f. $\sqrt{98}$

Exercise 2: Simplify each radical below

a. $\sqrt{x^2}$

b. $\sqrt{x^3}$

c. $\sqrt{x^4}$

d. $\sqrt{x^5}$

e. $\sqrt{x^6}$

f. $\sqrt{x^7}$

g. $\sqrt{x^{98}}$

h. $\sqrt{x^{99}}$

Write a rule for simplifying a radical of the form $\sqrt{x^n}$

When n is even, $\sqrt{x^n} \ =$

When n is odd, $\sqrt{x^n} \ =$

Exercise 3: Simplify each radical

a) $\sqrt{12x^6y^3}$

b) $3x\sqrt{20x^3y^6z^9}$

Helpful Radical Multiplication Rule:

Square of a Radical: $(\sqrt{a})^2 = (\sqrt{a})(\sqrt{a}) = a$

Product of radical conjugates: $(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b}) = (\sqrt{a})^2 - (\sqrt{b})^2 = a - b$

Strategy for rationalizing denominators with radicals:

If the denominator has one term, multiply the numerator and the denominator by the radical:

Example $\dfrac{2}{\sqrt{a}} = \dfrac{2}{\sqrt{a}} \cdot \dfrac{\sqrt{a}}{\sqrt{a}} = \dfrac{2\sqrt{a}}{a}$

If the denominator has two terms, multiply the numerator and the denominator by the conjugate of the denominator:

Examples a) $\dfrac{3}{\sqrt{a}+\sqrt{b}} = \dfrac{3}{\sqrt{a}+\sqrt{b}} \cdot \dfrac{\sqrt{a}-\sqrt{b}}{\sqrt{a}-\sqrt{b}} = \dfrac{3(\sqrt{a}-\sqrt{b})}{a-b}$

Exercise 4: Rationalize denominators. Simplify the answer.

a) $\frac{9}{\sqrt{7}}$

b) $\frac{3}{2\sqrt{x}}$

c) $\frac{-12}{\sqrt{10}-\sqrt{6}}$

d) $\frac{\sqrt{6}}{5-\sqrt{2}}$

Complex numbers

We define the imaginary unit $i = \sqrt{-1}$ as the solution to $x^2 = -1$.
The square root of a negative number can be simplified $\sqrt{-b} = \sqrt{-1}\sqrt{b} = i\sqrt{b}$ where $b > 0$

Exercise 5: Simplify the expressions.

a. $\sqrt{-81}$

b. $\sqrt{-75}$

c. $-\sqrt{-49}$

d. $\sqrt{-15}$

Powers of i :

$i^1 = i$	$i^5 =$ ________	$i^9 =$ ________
$i^2 = -1$	$i^6 =$ ________	$i^{10} =$ ________
$i^3 = -i$	$i^7 =$ ________	$i^{11} =$ ________
$i^4 = 1$	$i^8 =$ ________	$i^{12} =$ ________

What is the pattern for powers of i ? Write a rule for it ____

____

Exercise 6: Simplify the product or quotient in terms of i

a) $(\sqrt{-9})\,(\sqrt{16})$

b) $\frac{\sqrt{-36}}{\sqrt{-9}}$

c) $(\sqrt{-12})\,(\sqrt{-6})$

d) $\frac{\sqrt{36}}{\sqrt{-2}}$

Caution:

Correct work: $(\sqrt{-12})\,(\sqrt{-6}) = \;(i\sqrt{12})(i\sqrt{6}) = i^2\sqrt{72} = -6\sqrt{2}$

Incorrect work: $(\sqrt{-12})\,(\sqrt{-6}) \;=\; \sqrt{72} \;=\; 6\sqrt{2}$

Correct work: $\dfrac{\sqrt{36}}{\sqrt{-2}} = \dfrac{\sqrt{36}}{i\sqrt{2}} = \dfrac{\sqrt{36}}{i\sqrt{2}} \cdot \dfrac{i}{i} = \dfrac{i\sqrt{36}}{i^2\sqrt{2}} = \dfrac{i\sqrt{18}}{-1} = \;-3i\sqrt{2}$

Incorrect work: $\dfrac{\sqrt{36}}{\sqrt{-2}} = \;\sqrt{-18}\; = i\sqrt{18} = 3i\sqrt{2}$

A complex number is a number of the form $x + yi$ where x and y are real numbers. We say x is the **real part** of the complex number and yi is the **imaginary part** of the complex number.

We can add, subtract, multiply, and divide complex numbers following similar procedures and the operations of radicals if the radicands are positive. If the radicands are negative, we must first convert them to $\boldsymbol{i}$.

Exercise 7: Perform the indicated operation.

a) $(-5 + 9i) - (-2 + 3i)$

b) $(2 + 7i)(5 - 3i)$

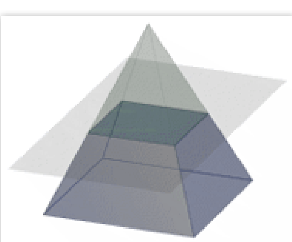
c) $(2 + 3i)(2 - 3i)$

The complex numbers $a + bi$ and $a - bi$ are called conjugates.
The product of complex Conjugates $(a + bi)(a - bi) \;=\; a^2 - (bi)^2 \;=\; a^2 + b^2$ is a real number.

Exercise 8: Rationalize the denominator.

a) $\dfrac{5+i}{2i}$

b) $\dfrac{3i}{5-2i}$



Complex number or imaginary number concept was first investigated by a mathematician and inventor named Heron (c. 10-70 A.D.) from the city of Alexandria on the coast of the Mediterranean, in Egypt. While trying to find the volume of the frustum of a pyramid (see Figure 1) with a square base of a certain size, Heron of Alexandria first encountered the square root of a negative number (Nahin, 1998).

Section 1.3 Methods for Solving Quadratic Equations

A quadratic function is of the form $f(x) = ax^2 + bx + c$, where a, b, c are real numbers and $a \neq 0$. The roots of the function are solutions to the equation $ax^2 + bx + c = 0$.

The quadratic equation $ax^2 + bx + c = 0$ can be solved in many ways:

I. By factoring and the Zero Factor Theorem
2. By the Square root property
3. By the Quadratic Formula

Method 1: Solving Quadratic Equations by Factoring

Zero Factor Theorem: If $ab = 0$ then $a = 0$ or $b = 0$ or both.

Solving Quadratic Equations by Factoring (Use this method when the equation is factorable)

1. **Put the equation in standard form: $ax^2 + bx + c = 0$.**
2. **Factor completely.**
3. **Use the zero-product rule, set each factor containing the variable equal to zero and solve for x. Note: Do not solve for the constant factor.**

Exercise 1: Solve the equations by factoring

a) Solve: $x^2 - 4x - 32 = 0$

b) Solve: $36 - 49y^2 = 0$

c) Solve: $a^3 - 6a^2 + 5a = 0$

d) Solve: $2x^3 + 9x^2 = 5x$

Method 2: Solving Quadratic Equations by the Square Root Property

The Square Root Property

If $x^2 = a$, then $x = \sqrt{a}$ or $x = -\sqrt{a}$.

Sometimes it is indicated as $x = \pm\sqrt{a}$

Solving Quadratic Equations by the Square Root Property (Use this method when the equation consists of a square term and a constant term)

- **Isolate the square and then apply the Square Root Property**

Exercise 1: Solve using the square root Property. Simplify the radical. Write out both solutions.

a) $x^2 = 100$

b) $s^2 - 12 = 0$

c) $s^2 + 27 = 0$

d) $(y-3)^2 - 98 = 0$

iv) $2(x+1)^2 + 48 = 0$

Method 3: Solving Quadratic Equations by the Quadratic Formula

The Quadratic Formula

The solutions to a quadratic equation of the form $ax^2 + bx + c = 0$, where $a \neq 0$ are given by the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Solving Quadratic Equations by the Quadratic Formula (This method can be used on any quadratic equations; it may not be the best method if the equation can be solved using the other two methods.)

1. **Put the equation in standard form: $\boldsymbol{ax^2 + bx + c = 0}$.**
2. **Substitute, $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ into the quadratic formula and simplify.**

Exercise 1: Solve using the quadratic formula

a) $x^2 - 5x - 8 = 0$

b) $x^2 + 2x + 6 = 0$

c) $2x^2 + 5 = 4x$

The **discriminant** is the radicand $b^2 - 4ac$ in the quadratic formula.

If the discriminant > 0, then the equation has two real roots.
If the discriminant = 0, then the equation has one real (double) root.
If the discriminant < 0, then the equation has two complex roots.

Exercise 2: Solve $4x^2 - 28x + 49 = 0$. What is the best method for solving this equation? Factoring, the square root property, or the quadratic formula? What type of roots does it have?

Exercise 3: Solve $x^2 + 63 = 0$. What is the best method for solving this equation? Factoring, the square root property, or the quadratic formula? What type of roots does it have?

Exercise 4: Solve $2x^2 - 6x - 9 = 0$. What is the best method for solving this equation? Factoring, the square root property, or the quadratic formula? What type of roots does it have?

SECTION 1.1 SUPPLEMENTARY EXERCISES

1. Factor completely

a) a^2-5a
b) $25y^3-15y^2$
c) $5ab^2-15a^3b$
d) $64w^2-81$
e) $36x^2-25y^2$
f) $16x-4x^3$
g) y^3-81y
h) $49-9t^2$
i) $36x^2y-9y$
j) $a^2-4a-12$
k) $x^2+9xy-36y^2$
l) m^3-4m^2-21m
m) $24+5n-n^2$
n) $2x^3-2x^2y-12xy^2$
o) $2x^3y+x^2y^2-6xy^3$
p) $-6x^2-9x+15$
q) $am-5a+2bm-10b$
r) $15x-12ax+10y-8ay$
s) $7b-2bd+21c-6cd$
t) $16a^5-250a^2$

SECTION 1.3 SUPPLEMENTARY EXERCISES

1. Solve the equations by factoring

a) $x^2+6x+8=0$
b) $2t^2+3t-14=0$
c) $x^2-10x+25=0$
d) $w^2+14w+55=6$
e) $t^2-5t=24$
f) $2y^3+12y^2-32y=0$
g) $k^2-25=0$
h) $16x^2=49$
i) $z^2-9z=0$
j) $-6x^2-x+12=0$
k) $5m^2+20m=6-9m$
l) $(y+5)^2-4=0$
m) $(n-3)(3n-2)-8n=0$
n) $10x^2=27x-18$

2. Solve the equations by the square root property

a) $x^2=120$
b) $s^2-45=0$
c) $(y+2)^2-49=0$
d) $\frac{1}{2}(x-6)^2-16=0$
e) $8=(b-5)^2-16$
f) $2k^2+80=0$
g) $(y-3)^2+64=0$
h) $y^2+88=7$
i) $(y+7)^2+50=0$
j) $(y-1)^2+12=0$
k) $(2k-1)^2+72=18$
l) $9(2m-3)^2+8=449$

3. Solve the equations by the quadratic formula

a) $p^2+5p=-2$
b) $a^2-2a=4$
c) $2k^2=-4k+3$
d) $12m+9m^2=-4$
e) $3x^2+6x+2=0$
f) $6x^2+2x+3=0$
g) $10x^2-13x-3=0$
h) $4x^2-x+6=0$
i) $m^2+2m=-5$
j) $2k^2+9k=-7$
k) $(z-2)(z+4)+6=0$
l) $(3x-7)(x+5)=-31$
m) $11n^2-4n(n-2)=6(n+3)$
n) $x(x-3)=-10x-7$

Module II
Polynomial Functions

Section 2.1 Definition of Functions, Domain, Range, and The Interval Notation

A function f is an assignment (or relation) that assigns to each input x in the domain exactly one output y in the codomain.

The domain of a function f is the set of all possible inputs of f. The codomain of a function is the set of all possible outputs of f. The range of a function f is a subset of codomain and the actual outputs of f.

For example,

$f: \mathbb{R} \to \mathbb{R}$ means the function f has a **domain** of $\mathbb{R}$ (all real numbers) and the **codomain** $\mathbb{R}$ (all real numbers).

If we define f as the assignment $f: x \mapsto x^2$, then f takes an input x and assigns it to the output x^2. We can also write it as

$$f(x) = x^2 \quad \text{or} \quad y = x^2$$

In this case, the **range** of f is the set of nonnegative real numbers, or $[0, \infty)$ using the interval notation.

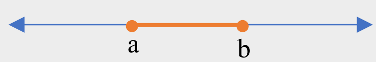
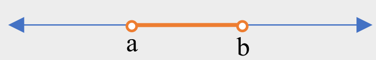
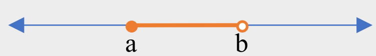
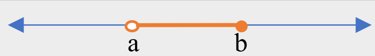
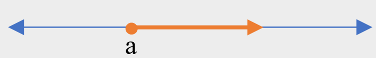
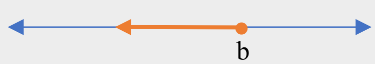
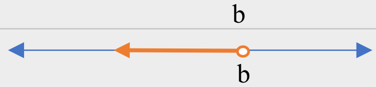
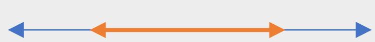
A function may assign the same output to two different inputs, but it may not assign two outputs to one input. In other words, two different inputs may have the same output, but one input may not have two different outputs.

Take the vending machine for example, one code (an input) returns one bottle of beverage. It won't return two bottles of beverages unless the machine malfunctions. On the other hand, it is possible that two different codes (different inputs) give us the same beverage (output).

Exercise 1. Determine if each of the following set forms a function. If it is a function, determine its domain.

	Function? (Yes or No)	Why or Why Not	If a function, what is its domain?
Set of ordered pairs { (1, 1), (2, 1), (3, 1) }			
Set of ordered pairs { (1, 1), (1, 2), (1, 3) }			
Fahrenheit-Celsius conversion formula $F = \frac{9}{5}C + 32$			
Equation of a line: $y = mx + b$			
Parabola: $y = ax^2 + bx + c$			
Circle: $x^2 + y^2 = 1$			

Interval on the real number line

Inequality notation	Real number line	Interval notation
$a \le x \le b$		$[a, b]$
$a < x < b$		(a, b)
$a \le x < b$		$[a, b)$
$a < x \le b$		$(a, b]$
$x \ge a$		$[a, \infty)$
$x > a$		(a, ∞)
$x \le b$		$(-\infty, b]$
$x < b$		$(-\infty, b)$
All real numbers		$(-\infty, \infty)$

Section 2.2 Quadratic Functions: Graphs and Roots

A **quadratic function** is of the form $f(x) = ax^2 + bx + c$, where a, b, c are real numbers and $a \neq 0$.

The graph of a quadratic function is a parabola (U-shaped). A parabola has either a maximum or a minimum point, called the **vertex**. The **axis of symmetry** for the graph of a quadratic function is the vertical line through the vertex.

The vertex can be found by the following methods:

Method 1: The Vertex Form
A quadratic function expressed as $f(x) = a(x-h)^2 + k$ is said to be in the vertex form, where (h, k) is the vertex.

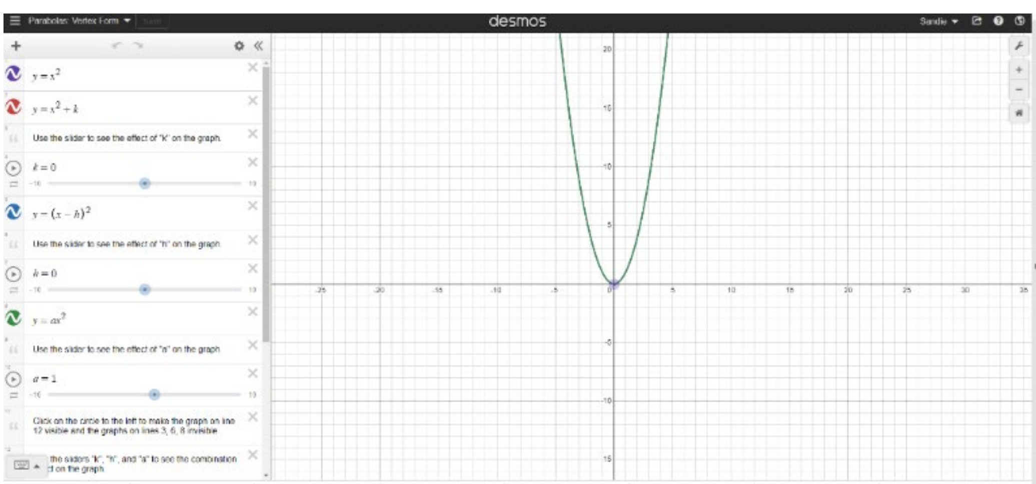
Method 2: The Vertex Formula
If a quadratic function is in standard form $f(x) = ax^2 + bx + c$, the vertex can be computed using the formula:

x-coordinate of the vertex is $-\frac{b}{2a}$

Substitute x-coordinate to find the y-coordinate of the vertex

The **roots** of a quadratic function are those x values such that $f(x) = 0$, or solutions to the equation $ax^2 + bx + c = 0$. They are also called the **zeros** of the function. The quadratic equation $ax^2 + bx + c = 0$ can be solved by factoring, by the square root property, or by the quadratic formula. See section 1.3 for methods of solving quadratic equations.

Use the Desmos link below to see the effect of changing h, k and a on the parabola.
https://www.desmos.com/calculator/0txid19ts5



How does the graph change when you change h? When h is positive? When h is negative?

____

____

How does the graph change when you change k? When k is positive? When k is negative?

____

____

How does the graph change when you change a? When a is positive? When a is negative? When $|a| > 1$? When $|a| < 1$?

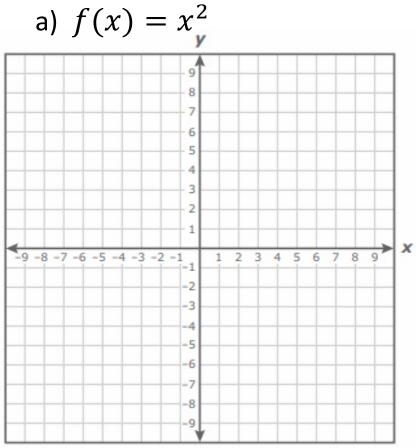
____

____

Exercise 1: For each quadratic function:

i) Identify the vertex.
ii) Sketch the graph.
iii) Determine the domain and the range of the function.
iv) Find the roots. Determine whether the roots are two real roots, one real root, or two complex roots.

a) $f(x) = x^2$



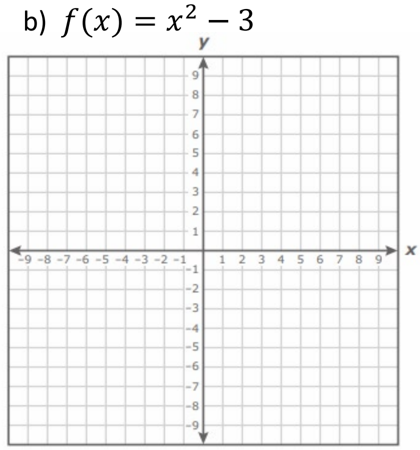
The vertex is ______________

The Domain is _____________

The Range is _______________

The roots are ____________________

b) $f(x) = x^2 - 3$



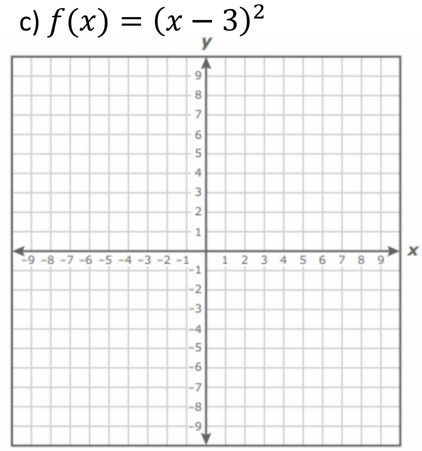
The vertex is ______________

The Domain is _____________

The Range is _______________

The roots are ____________________

c) $f(x) = (x - 3)^2$



The vertex is ______________

The Domain is _____________

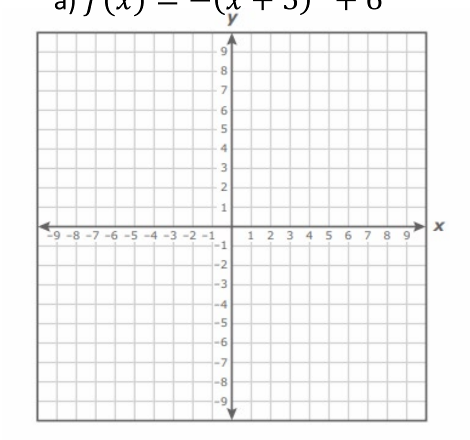
The Range is _______________

The roots are ____________________

Exercise 2: For each quadratic function,

i. Use the **Vertex Form** to determine the vertex.
ii. Sketch the graph.
iii. Determine the domain and the range of the function.
iv. Find the roots. Determine whether the roots are two real roots, one real root, or two complex roots.

a) $f(x) = -(x+5)^2 + 6$



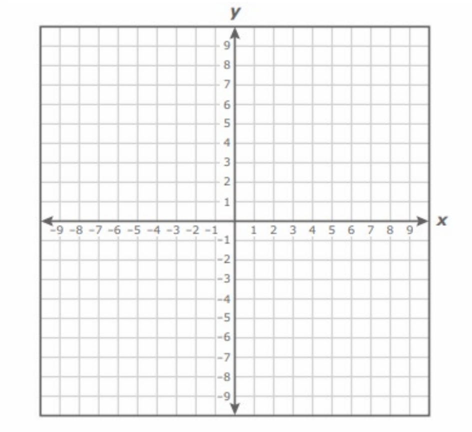
The vertex is _______________

The Domain is ______________

The Range is ________________

The roots are _____________________

b) $f(x) = 2(x+2)^2 - 8$



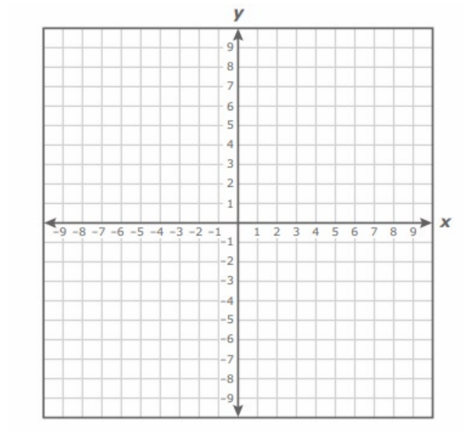
The vertex is _______________

The Domain is ______________

The Range is ________________

The roots are _____________________

c) $f(x) = -\frac{1}{2}x^2 - 2$



The vertex is _______________

The Domain is ______________

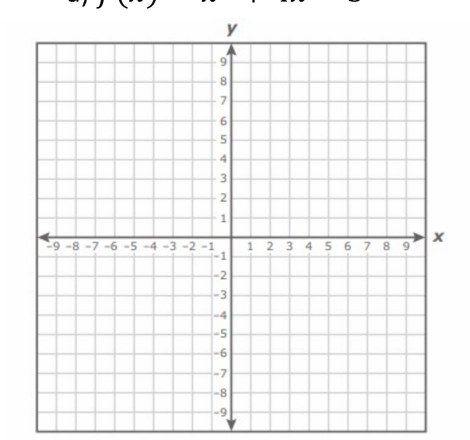
The Range is ________________

The roots are _____________________

Exercise 3: For each quadratic function,

i. Use the **Vertex Formula** to determine the vertex.
ii. Sketch the graph.
iii. Determine the domain and the range of the function.
iv. Find the roots. Determine whether the roots are two real roots, one real root, or two complex roots.
v. Determine the interval where the function is positive or negative

a) $f(x) = x^2 + 4x - 5$



The vertex is _______________

The Domain is ______________

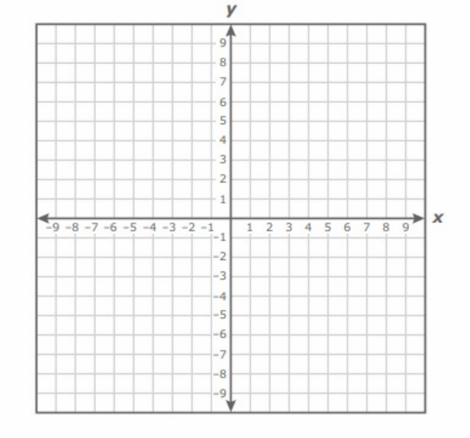
The Range is ________________

The roots are ____

The function is positive on the interval __________________

The function is negative on the interval __________________

b) $f(x) = -x^2 - 2x - 5$



The vertex is ______________

The Domain is _____________

The Range is _______________

The roots are _____________________________________

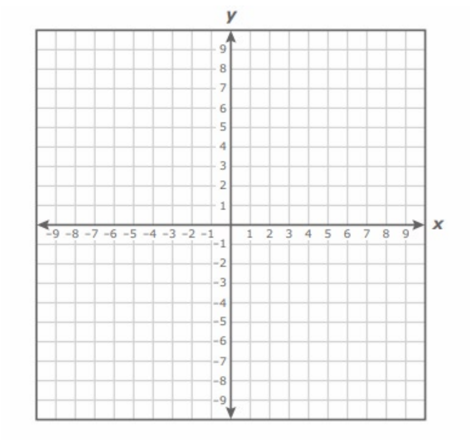
The function is positive on the interval _________________

(use decimal approximation)

The function is negative on the interval _________________

(use decimal approximation)

c) $f(x) = x^2 - 2x + 5$



The vertex is ______________

The Domain is _____________

The Range is _______________

The roots are _____________________________________

The function is positive on the interval _________________

The function is negative on the interval _________________

How do you identify the real roots of the function? ____

____

What do you notice about the roots in exercises 2f and 3c? How are their graphs different from the rest? What can you say about the graph of a quadratic function with complex roots?

____

____

Section 2.3 Polynomial Functions: Graphs and Roots

A polynomial function is a function of the form $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ where $a_n \neq 0$, the exponents are all whole numbers, and the coefficients are all real numbers. a_n is the leading coefficient, and a_0 is the constant term, and n is the degree of the polynomial.

The **roots** of a polynomial function are those x values such that $f(x) = 0$. They are also called the **zeros** or **solutions** of the polynomial function.

A polynomial function has the following characteristics:

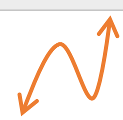
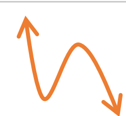
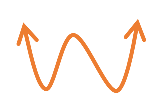
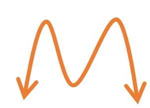
- A polynomial function with degree n has n roots (including multiplicity), they may be real or complex roots.
- A polynomial function may cross the x-axis at most n times, corresponding to at most n distinct real roots.
- If a polynomial function with real coefficients has complex roots, the complex roots always exist in conjugate pair.
- A polynomial function may have a graph with at most $n - 1$ turning points.

The END behavior

The END behavior of the graph of a polynomial function

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$$

Is determined by n and a_n , where n is the degree of the polynomial and a_n is the leading coefficient of the polynomial.

Degree of the polynomial	Sign of the Leading Coefficient	Left End Behavior As $x \to -\infty$	Right End Behavior As $x \to \infty$	A maximum or minimum exists?	Domain & Range	Example
n is odd	a_n is positive	$f(x) \to -\infty$ or Down	$f(x) \to \infty$ or Up	No	D = $(-\infty, \infty)$ R = $(-\infty, \infty)$	
n is odd	a_n is negative	$f(x) \to \infty$ or Up	$f(x) \to -\infty$ or Down	No	D = $(-\infty, \infty)$ R = $(-\infty, \infty)$	
n is even	a_n is positive	$f(x) \to \infty$ or Up	$f(x) \to \infty$ or Up	There is a minimum, **m**	D = $(-\infty, \infty)$ R = $[\mathbf{m}, \infty)$	
n is even	a_n is negative	$f(x) \to -\infty$ or Down	$f(x) \to -\infty$ or Down	There is a maximum, **M**	D = $(-\infty, \infty)$ R = $(-\infty, \mathbf{M}]$	

Synthetic Division

The method of Synthetic Division is a quick way to divide a polynomial $f(x)$ by a linear factor of the form $(x - k)$. Instead of using long division, we can simply work with the coefficients of $f(x)$ and k.

Example: If $f(x) = ax^2 + bx + c$. Divide $f(x)$ by $(x - k)$.

Step 1: Set up synthetic division using the coefficients a, b, c and k.

k	a	b	c

Step 2: Bring down the first coefficient a. Multiply a by k and add to b

k	a	b	c
		ka	
	a	$b + ka$	

Step 3: Repeat the procedure. Bring down $b + ka$. Multiply $b + ka$ by k, and add to c.

k	a	b	c
		ka	$k(b + ka)$
	a	$b + ka$	$c + k(b + ka)$

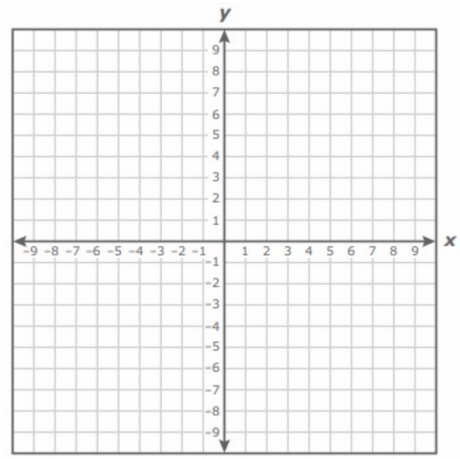
Step 4: Repeat the procedure until we finish the last coefficient. The result of the division is a quotient polynomial $q(x) = ax + (b + ka)$ with remainder $c + k(b + ka)$. Note: The degree of the quotient is 1 less than the degree of the original polynomial.

k	a	b	c	
		ka	$k(b + ka)$	
Coefficients of $q(x)$	a	$b + ka$	$c + k(b + ka)$	Remainder

Exercise 1. If $f(x) = 3x^2 - 8x + 17$ is divided by $x - 4$. Determine the quotient and the remainder.

Exercise 2. If $f(x) = x^3 - 2x^2 - 5x + 6$.

a) Divide $f(x)$ by $x + 2$ using synthetic division. Determine the quotient and the remainder. What is the significance of the remainder being zero?
b) Set the quotient to zero and solve.
c) Express f in factored form.
d) Find all three roots of f. What is the characteristic of the roots? (Rational, irrational, complex?)
e) Sketch a graph of f, check the graph on Desmos or graphing calculator. How do the roots of the function relate to its graph?



The Remainder and Factor Theorems

The Remainder Theorem:

If a polynomial $f(x)$ is divided by $x - k$, then the remainder is $f(k)$.

The Factor Theorem:

If a polynomial $f(x)$ has a factor $x - k$, if and only if $f(k) = 0$.

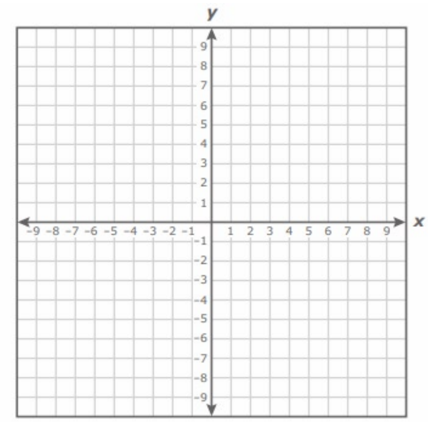
(i.e., if the remainder is zero, then the polynomial is factorable.)

The following are equivalent:

- r **is a root of** $f(x)$
- r **is a zero of** $f(x)$
- r **is a solution of** $f(x)$
- $x - r$ **is a factor of** $f(x)$
- $f(r) = 0$
- **The remainder of** $f(x)$ **divided by** $x - k$ **is zero**

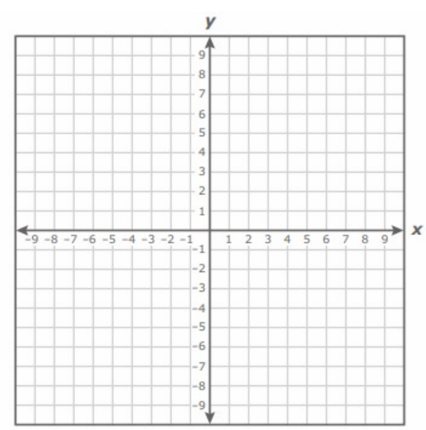
Exercise 3. If $f(x) = x^3 + 2x^2 + 4x + 8$

a) Find a rational root r of the polynomial function using Desmos or graphing calculator.
b) Use synthetic division to divide $f(x)$ by $x - r$.
f) Find the remaining two roots by solving the quadratic function. Express f in factored form. What is the characteristic of the roots? (Multiplicity?)
c) Sketch a graph. How do the roots of the function relate to its graph?



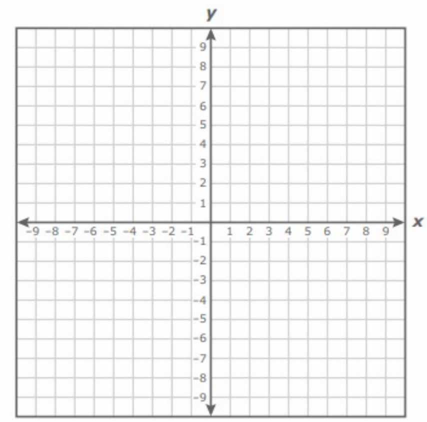
Exercise 4. If $f(x) = x^3 - x^2 - 10x + 12$

a) Find a rational root r of the polynomial function using Desmos or graphing calculator.
b) Use synthetic division to divide $f(x)$ by $x - r$.
c) Find the remaining two roots by solving the quadratic function. What is the characteristic of the roots? (Rational, irrational, complex?)
d) Sketch a graph. How do the roots of the function relate to its graph?



Exercise 5. If $f(x) = x^4 + x^3 - x^2 + x - 2$

a) Find two rational roots r_1 and r_2 of the polynomial function using Desmos or graphing calculator.
b) Apply synthetic division twice: First divide $f(x)$ by $x - r_1$, then divide the resulting quotient by $x - r_2$.
c) Find the remaining two roots by solving the quadratic function. What is the characteristic of the roots? (Rational, irrational, complex?)
d) Sketch a graph. How do the roots of the function relate to its graph?



Exercise 6. Find the polynomial of degree 3 with roots 1, 2, -3, and $f(-1) = 8$.

a) Express the polynomial in factored form $f(x) = a(x - r_1)(x - r_2)(x - r_3)$.
b) Use $f(-1) = 8$ to solve for a.

Exercise 7. Find a polynomial of degree 4 with real coefficients and roots $2i$ and 3. Root 3 has multiplicity 2.

SECTION 2.2 SUPPLEMENTARY EXERCISES

For each quadratic function,

i. Determine the vertex.
ii. Sketch the graph.
iii. Determine the domain and the range of the function.
iv. Find the roots. Determine whether the roots are two real roots, one real root, or two complex roots.

a) $f(x) = (x-3)^2 - 1$

b) $f(x) = -(x-1)^2 + 7$

c) $f(x) = \frac{1}{3}(x-8)^2 - 7$

d) $f(x) = -(x-2)^2 + 9$

e) $f(x) = 6(x+5)^2 + 4$

f) $f(x) = \frac{1}{2}(x+5)^2 - 3$

g) $f(x) = -(x-3)^2 - 1$

h) $f(x) = 4(x+7)^2 + 8$

i) $f(x) = 2x^2 + 4x - 5$

j) $f(x) = -x^2 + 4x - 6$

SECTION 2.3 SUPPLEMENTARY EXERCISES

1. a) Divide $x^3 + 2x^2 - 6x - 9$ by $x-2$

   b) Divide $x^3 + 2x^2 - 6x - 9$ by $x+3$

   c) Divide $2x^3 - 3x^2 - 5x + 18$ by $x+2$

2. Find the zeros of a function

   a) One zero of $f(x) = x^3 - 2x^2 - 9x + 18$ is $x = 2$. Find the other zeros.

   b) One zero of $f(x) = 4x^3 + 9x^2 - 52x + 15$ is $x = -5$. Find the other zeros.

   c) Find all the real zeros for $f(x) = x^4 + 4x^3 - 9x^2 - 36x$.

   d) Find all the real zeros for $f(x) = x^4 - 13x^2 + 36$.

3. Find a polynomial of degree 4 with real coefficients and roots i and $2+i$.

Module III

Exponential and Logarithmic Functions

Section 3.1 Exponential Functions and Graphs

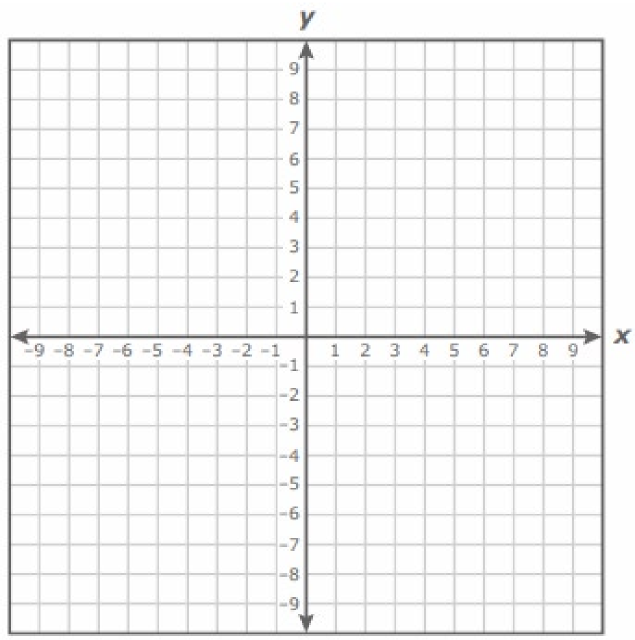
For any real number x, an exponential function is a function of the form

$$f(x) = c \cdot b^x$$

Where the base b is a positive real number such that $b \neq 1$ and c is a non-zero real number.

Example 1: Graph the exponential function: $f(x) = 2^x$. What is its domain and range?

x	$y = f(x)$
-3	
-2	
-1	
0	
1	
2	
3	



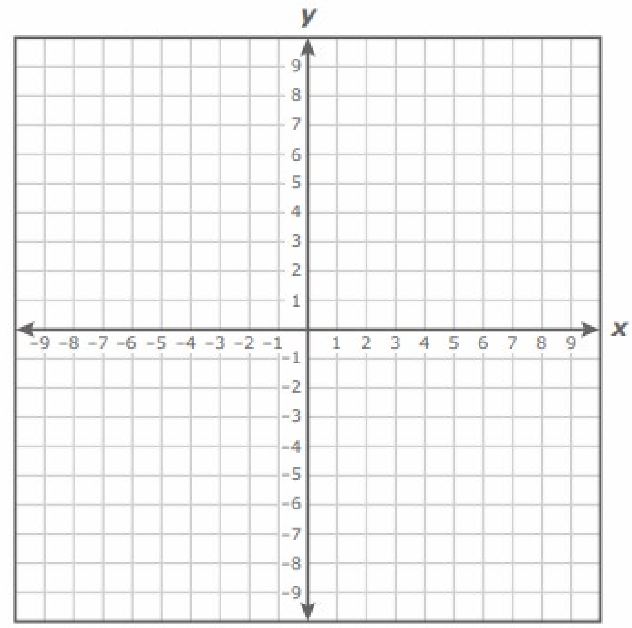
Characteristics of the exponential graph: $y = c \cdot b^x$

- The exponential graph is always increasing if $b > 1$; it is always decreasing if $0 < b < 1$.
- The graph does not cross the x-axis; $y = 0$ is a horizontal asymptote.
- The graph has the y-intercept $(0,1)$.
- The domain of the exponential function is all real numbers $(-\infty, \infty)$.
- If $c > 0$, the range of the exponential function is $(0, \infty)$. If $c < 0$, the range is $(-\infty, 0)$.

Example 2:

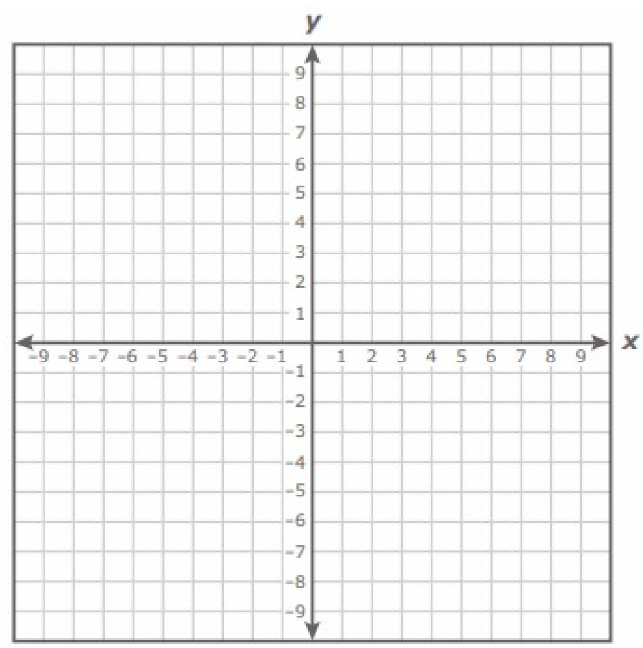
a) Graph the exponential function: $g(x) = \left(\frac{1}{2}\right)^x$. What is its domain and range?
b) Graph the exponential function: $h(x) = 2^{-x}$. What is its domain and range?
c) How do the two graphs compare? Why is it so?

x	$g(x) = \left(\frac{1}{2}\right)^x$	$h(x) = 2^{-x}$
-3		
-2		
-1		
0		
1		
2		
3		



Example 3: Graph the exponential function: $k(x) = -2^x$. What is its domain and range?

x	$y = k(x)$
-3	
-2	
-1	
0	
1	
2	
3	



Section 3.2 Logarithmic Functions and Graphs

If $b > 0$ and $b \neq 1$, then $y = \log_b x \quad \Leftrightarrow \quad x = b^y$

Exercise 1: Evaluate the logarithms and rewrite each logarithmic relation as an exponential relation.

Evaluate the logarithm	**Rewrite in exponential form**
$\log_3 9 = ?$	$\log_3 9 = 2 \quad \Leftrightarrow \quad 3^2 = 9$
$\log_3 1$	
$\log_3 \frac{1}{9} =$	
$\log_2 8 =$	
$\log_8 2 =$	
$\log_{1/2} 2 =$	

Exercise 2: Evaluate the logarithms.

a) $\log_{10} 1$

b) $\mathrm{Log}_{10} 10$

c) $\log_{10} 100$

d) $\log_{10} 1000$

e) $\log_{10}\left(\frac{1}{10}\right)$

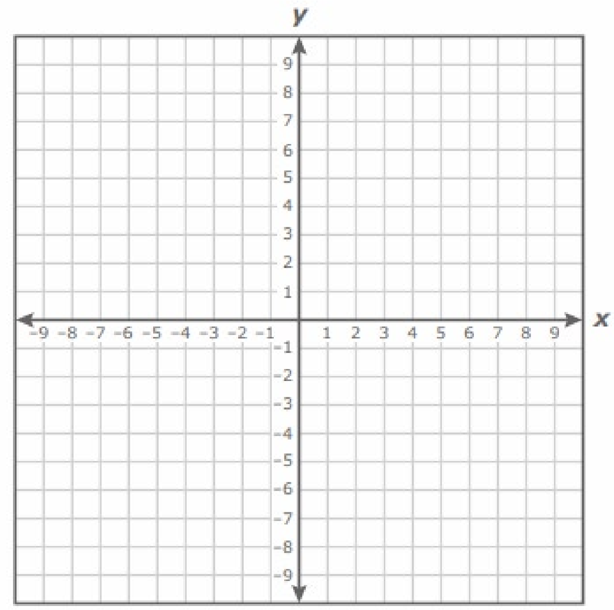
f) $\log_{10}\left(\frac{1}{100}\right)$

g) $\log_{10}\left(\frac{1}{1000}\right)$

Exercise 3: Graph the logarithmic function: $y = \log_2 x$.
Convert logarithm to exponential form; complete the table of values and graph. What is its domain and range?

$y = \log_2 x \iff x = 2^y$

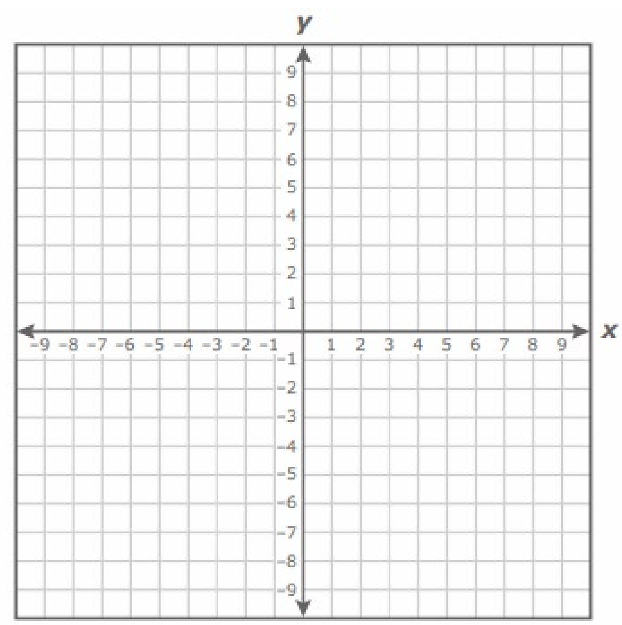
x	y
	-3
	-2
	-1
	0
	1
	2
	3



Exercise 4: Graph the logarithmic function: $y = \log_3 x$.
Convert logarithm to exponential form; complete the table of values and graph. What is its domain and range?

$y = \log_3 x \iff x = 3^y$

x	y
	-2
	-1
	0
	1
	2



Characteristics of the logarithmic function: $f(x) = \log_b x$

- The function does not cross the y-axis; $x = 0$ is a vertical asymptote.
- The function has the x-intercept $(1, 0)$.
- The domain of the exponential function is $(0, \infty)$.
- The range of the exponential function is $(-\infty, \infty)$.

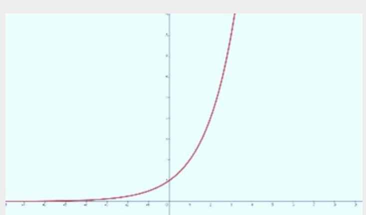
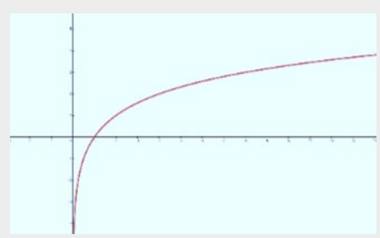
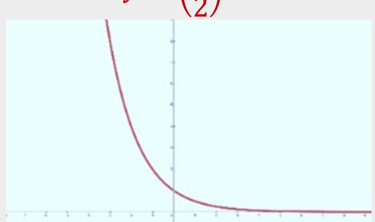
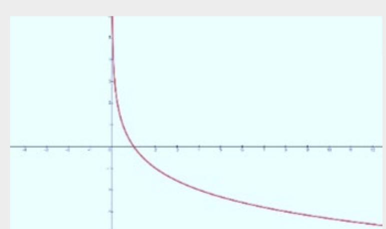
Use the link to try the Desmos activity: https://www.desmos.com/calculator/bvh8odgu6n
How do the exponential and logarithmic graphs change when the base is >1? When the base is between 0 and 1? ____

____

What is the inverse symmetry between the exponential and the logarithmic functions of the same base?

____

____

Comparing the Graphs of $y = b^x$ and $y = \log_b x$ $(b > 0,\ b \neq 1)$		
	$y = b^x$	$y = \log_b x$
Domain	$(-\infty, \infty)$	$(0, \infty)$
Range	$(0, \infty)$	$(-\infty, \infty)$
x-intercept	None	$(1, 0)$
y-intercept	$(0, 1)$	None
Contains points	$(0,1), (1, b)$	$(1, 0), (b, 1)$
Asymptote	x-axis or horizontal line $y = 0$	y-axis or vertical line $x = 0$
Example $b > 1$ (Always increasing)	$y = 2^x$ 	$y = \log_2 x$ 
Example $0 < b < 1$ (Always decreasing)	$y = \left(\frac{1}{2}\right)^x$ 	$y = \log_{1/2} x$ 

Section 3.3 Properties of Logarithms

Properties of Logarithms

	Properties of Exponents	Corresponding Properties of Logarithms
Product Property	$b^{x+y} = b^x \cdot b^y$	$\log_b(x \cdot y) = \log_b x + \log_b y$
Quotient Property	$b^{x-y} = \frac{b^x}{b^y}$	$\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$
Power Property	$(b^x)^n = b^{nx}$	$\log_b(x^n) = n \cdot \log_b x$
	$b^1 = b$	$\log_b(b) = 1$
	$b^0 = 1$	$\log_b(1) = 0$

Exercise 1: Use the properties of logarithm and the values $\log_5 3 \approx 0.683$ and $\log_5 7 \approx 1.209$ to evaluate each of the following. (What is the value of $\log_5 5$?)

a) $\log_5 21$

b) $\log_5 \frac{3}{7}$

c) $\log_5 35$

d) $\log_5 \frac{15}{7}$

e) $\log_5 49$

The logarithm with base 10 is called the **common logarithm** denoted by $\log_{10}$ or $\log$

$$\log_{10} x = \log x$$

The logarithm with base *e* is called the **natural logarithm** denoted by $\log_e$ or $\ln$

$$\log_e x = \ln x$$

The Number *e*

The **natural base *e*** is a mathematical constant defined as the value $\left(1+\frac{1}{n}\right)^n$ as n approaches ∞. Its value is approximately

$$e \approx 2.71828182845$$

Properties of log and ln

log	ln
$\log(x \cdot y) = \log x + \log y$	$\ln(x \cdot y) = \ln x + \ln y$
$\log\left(\frac{x}{y}\right) = \log x - \log y$	$\ln\left(\frac{x}{y}\right) = \ln x - \ln y$
$\log x^n = n \log x$	$\ln x^n = n \ln x$
$\log 10 = 1$	$\ln e = 1$
$\log 1 = 0$	$\ln 1 = 0$

Exercise 2: Expand using the properties of logarithms. Assume x and y are positive.

a) $\log_3\left(\frac{9\sqrt{x}}{y^2}\right)$

b) $\log\frac{(x+3)}{100}$

c) $\ln\left(\frac{6e^4}{4}\right)$

Exercise 3: Condense into one single logarithm.

a) $\log 6 + 2\log 2 - \log 3$

c) $9\log a - 4\log 2$

e) $7\ln x + 6\ln(x+4) - 3\ln y$

The inverse relation of a logarithmic and exponential functions:

$$\log_b b^x = x$$

$$b^{\log_b x} = x$$

The inverse relation in terms of the common log and the natural log:

$$\log 10^x = x \qquad \ln e^x = x$$

$$10^{\log x} = x \qquad e^{\ln x} = x$$

Exercise 4: Simplify the expressions. Assume x and y are positive.

a) $10^{\log 2^3}$

b) $\log_3 9^x$

c) $\ln e^{(3x+y)}$

d) $e^{2\ln x}$

4.3 Solving Exponential and Logarithmic Equations

Let $b > 0$ and $b \neq 1$. Then

$$b^x = b^y \quad \Leftrightarrow \quad x = y.$$

$$\log_b x = \log_b y \quad \Leftrightarrow \quad x = y$$

Solving Exponential Equations

If it's possible to write both sides of the equation as a power of the **same** base, use the relation $b^x = b^y \quad \Leftrightarrow \quad x = y$ to set the exponents equal to each and solve.

Example 1: Solve $9^x = 27$

$$(3^2)^x = (3)^3$$

$$(3)^{2x} = (3)^3$$

$$\Rightarrow \quad 2x = 3$$

$$\Rightarrow \quad x = \frac{3}{2}$$

If it's **not** possible to write both sides of the equation as a power of the **same** base, use the relation $\log x = \log y \quad \Leftrightarrow \quad x = y$ and the power property $\log x^n = n \log x$ to apply **log** on both sides of the equation, simplify, and solve.

Example 2: Solve $9^x = 26$

$$\log 9^x = \log 26$$

$$x \cdot \log 9 = \log 26$$

$$\Rightarrow \quad x = \frac{\log 26}{\log 9} \approx 1.4828$$

Repeat Example 2, this time, apply **ln** instead of **log**:

$$\ln 9^x = \ln 26$$

Use the calculator to find the decimal approximation. How does the answer compare with above?

Exercise 1: Solve by rewriting both sides of the equation as a power of the **same** base

a) Solve: $2^{x+1} = 8$

b) Solve: $16^{x+1} = 64$

c) Solve: $\left(\frac{1}{6}\right)^x = 36$

d) Solve: $4^{2x+3} = 8^{x+2}$

Exercise 2: Solve by applying either **log** or **ln** on both sides of the equation. Round answer to the nearest ten thousandths (four decimal places).

a) Solve $2^x = 5$

b) Solve $41^x = 25$

Solving Logarithmic Equations

First, use properties of logarithms to combine log terms or simplify to one of the following forms:

1. If **log** appears only on one side of the equation in the form of $\log_b x = y$, rewrite it as an exponential equation $b^y = x$ and solve.

2. If **log** appears on both sides of the equation in the form of $\log_b x = \log_b y$, set $x = y$ and solve.

Exercise 3: Solve each logarithmic equation

a) Solve: $\log_6 x = 2$

b) Solve: $\log_x 32 = 5$

c) Solve: $\log_7 23 = x$

d) Solve: $\log_3(5x - 1) = \log_3(x + 7)$

e) Solve: $\log x + \log(x - 15) = 2$ **Remember to check for extraneous solutions!**

SECTION 3.3 SUPPLEMENTARY EXERCISES

1. Expand using the properties of logarithms. Assume x and y are positive.
   a) $\log_2 \frac{7x^3}{y}$
   b) $\log_5 \frac{5a^8}{b^3}$

2. Condense into a single logarithm.
   a) $5\log x + 2\log y - 4\log z$
   b) $\frac{1}{3}\log w + \frac{1}{2}\log p - 3\log 5$

SECTION 3.4 SUPPLEMENTARY EXERCISES

1. Solve for x
   a) Solve: $4^x = 32$
   b) Solve: $27^{x-2} = 9^{x+1}$
   c) Solve: $3^x = 67$
   d) Solve: $58^x = 38$

2. Evaluate and round answer to the nearest ten thousandths (four decimal places).
   i. Set the expression equal to x. Rewriting it as an exponential equation and solve.
   ii. Or use Change of Base Formula: $\log_b x = \frac{\log x}{\log b}$ or $\log_b x = \frac{\ln x}{\ln b}$
   a) $\log_3 7$
   b) $\log_2 6$
   c) $\log_8 5$

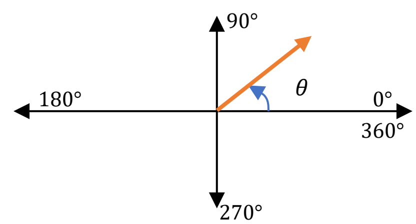
3. Solve the logarithmic equation. **Remember to check for extraneous solutions!**
   a) Solve: $\log_5(3x+1) = 2$
   b) Solve: $\log_x\left(\frac{1}{64}\right) = 3$
   c) Solve: $\log_6 x + \log_6(x+1) = 2$
   d) Solve: $\log 5x + \log(x-1) = 2$
   e) Solve: $2 - \log(x-1) = \log(x-16)$

Module IV
Trigonometric Functions

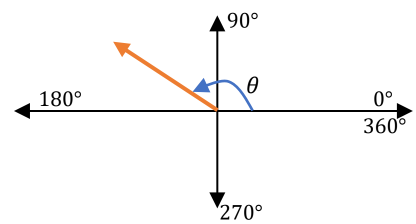
Section 4.1 The Unit Circle

An angle θ in standard form has its initial side on the positive x-axis (measured counterclockwise from 0°) to the terminal side of θ.

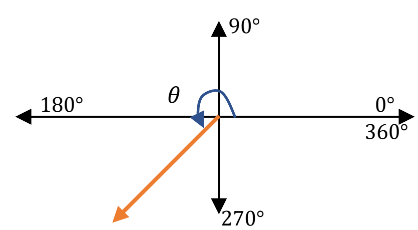
Quadrant I contains angles between ______________________



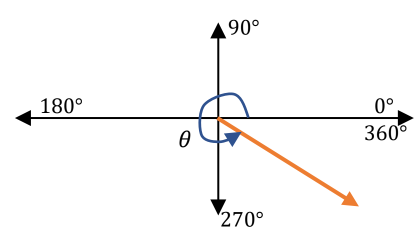
Quadrant II contains angles between ______________________



Quadrant III contains angles between ______________________



Quadrant IV contains angles between ______________________



Converting from Degrees to Radians and vice versa

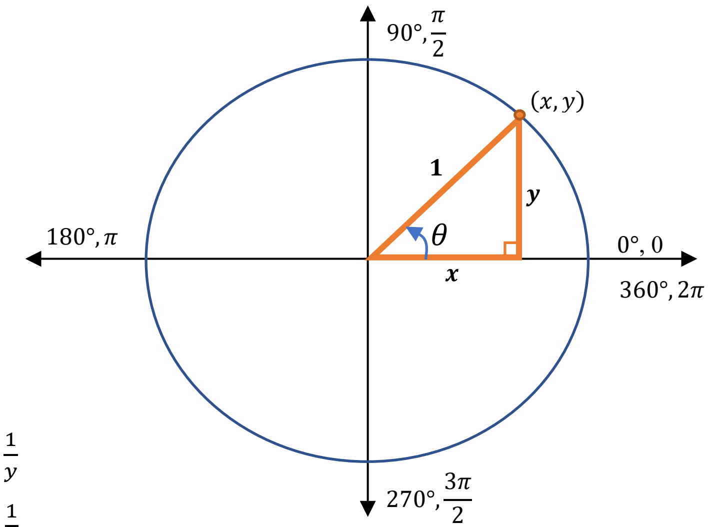
To change from degrees to radians- multiply the degree by $\frac{\pi}{180}$

To change from radians to degrees- multiply the radian by $\frac{180}{\pi}$

Exercise 1: Find the corresponding degree or radian

Degree	Radian
$30°$	
$45°$	
	$\frac{\pi}{3}$
	$\frac{\pi}{2}$
$120°$	
	$\frac{3\pi}{4}$
	$\frac{5\pi}{6}$
	π
$210°$	
	$\frac{5\pi}{4}$
$240°$	
$270°$	
	$\frac{5\pi}{3}$
$315°$	
$330°$	
$360°$	

Defining Trigonometric functions on the Unit Circle:



$\sin\theta = y$ $\qquad$ $\csc\theta = \frac{1}{y}$

$\cos\theta = x$ $\qquad$ $\sec\theta = \frac{1}{x}$

$\tan\theta = \frac{y}{x}$ $\qquad$ $\cot\theta = \frac{x}{y}$

The Reciprocal Identities

$$\sin\theta = \frac{1}{\csc\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

$$\cos\theta = \frac{1}{\sec\theta} \qquad \sec\theta = \frac{1}{\cos\theta}$$

$$\tan\theta = \frac{1}{\cot\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

The Quotient Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Quadrant II $90° < \theta < 180°$ or $\frac{\pi}{2} < \theta < \pi$

S **Sine is positive**

$\sin\theta$, $\csc\theta$ are positive
$\cos\theta$, $\sec\theta$, $\tan\theta$, $\cot\theta$ are negative

Quadrant I $0° < \theta < 90°$ or $0 < \theta < \frac{\pi}{2}$

A **All are positive**

$\sin\theta$, $\csc\theta$, $\cos\theta$, $\sec\theta$, $\tan\theta$, $\cot\theta$ are all positive

Quadrant III $180° < \theta < 270°$ **or** $\pi < \theta < \frac{3\pi}{2}$

T **Tangent is positive**

$\tan\theta$, $\cot\theta$ are positive
$\sin\theta$, $\csc\theta$, $\cos\theta$, $\sec\theta$ are negative

Quadrant IV $270° < \theta < 360°$ or $\frac{3\pi}{2} < \theta < 2\pi$

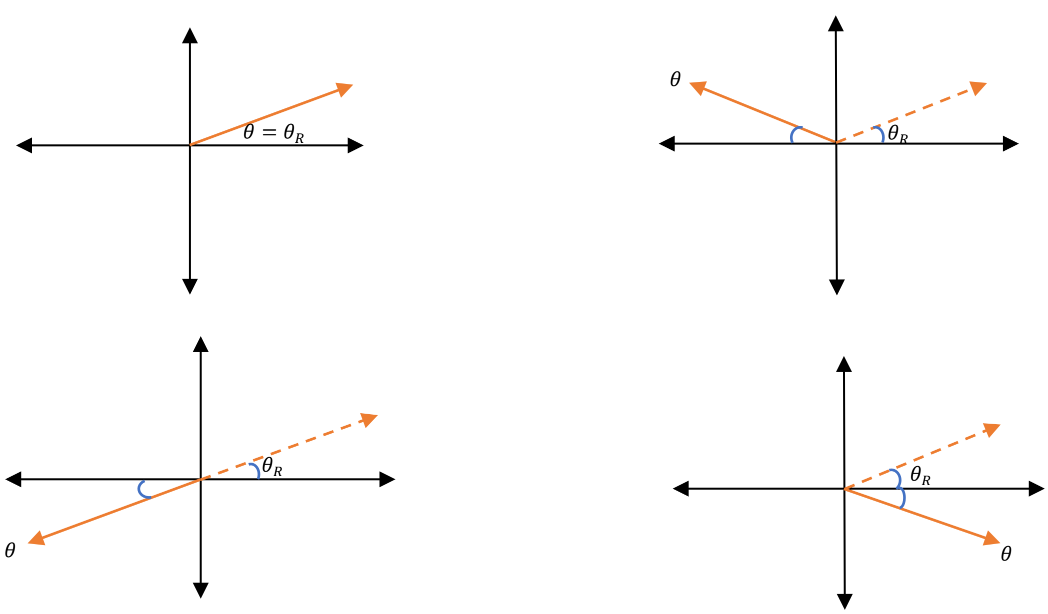
C **Cosine is positive**

$\cos\theta$, $\sec\theta$ are positive
$\sin\theta$, $\csc\theta$, $\tan\theta$, $\cot\theta$ are negative

Reference Angle

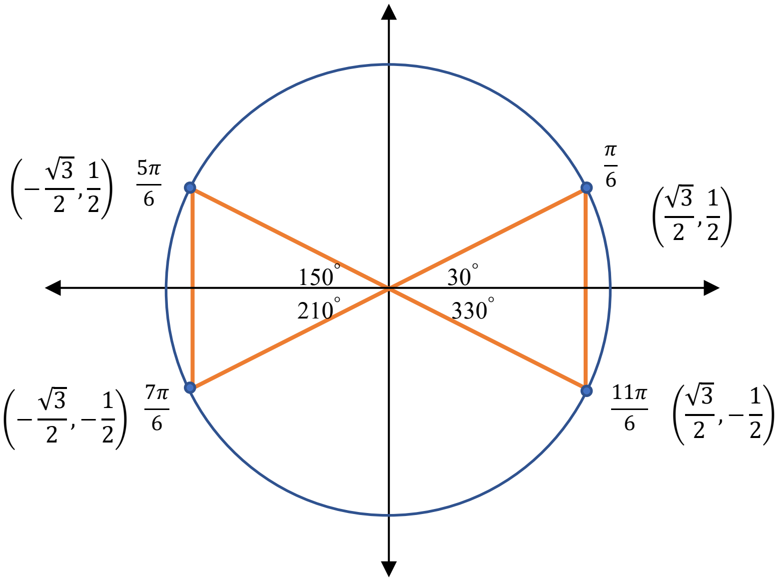
The **reference angle** θ_R for any angle θ in standard position is the positive acute angle between the terminal side of θ and the x-axis.

Angle and its reference angle:



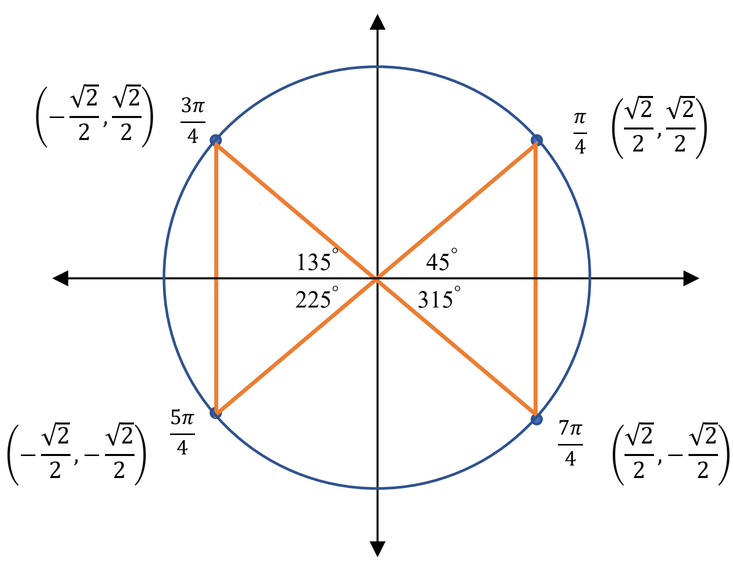
Use a bow-tie to relate angles with 30° or $\frac{\pi}{6}$ reference angle.

What can you say about their coordinates on the unit circle? ____________________________________



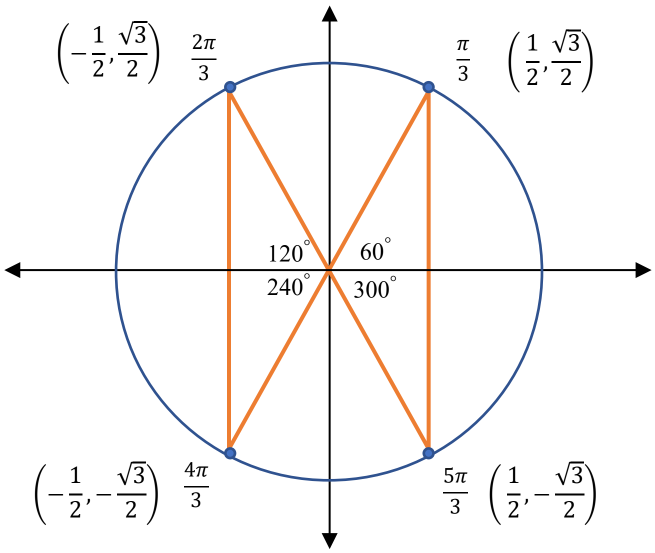
Use a bow-tie to relate angles with 45° or $\frac{\pi}{4}$ reference angle

What can you say about their coordinates on the unit circle? ____________________________________



Use a bow-tie to relate angles with 60° or $\frac{\pi}{3}$ reference angle

What can you say about their coordinates on the unit circle? ___________________________________



Try both Desmos activities below to see how the coordinates on the unit circle change with angles.

Desmos: The Unit Circle 1
https://www.desmos.com/calculator/bydhfz5b9d

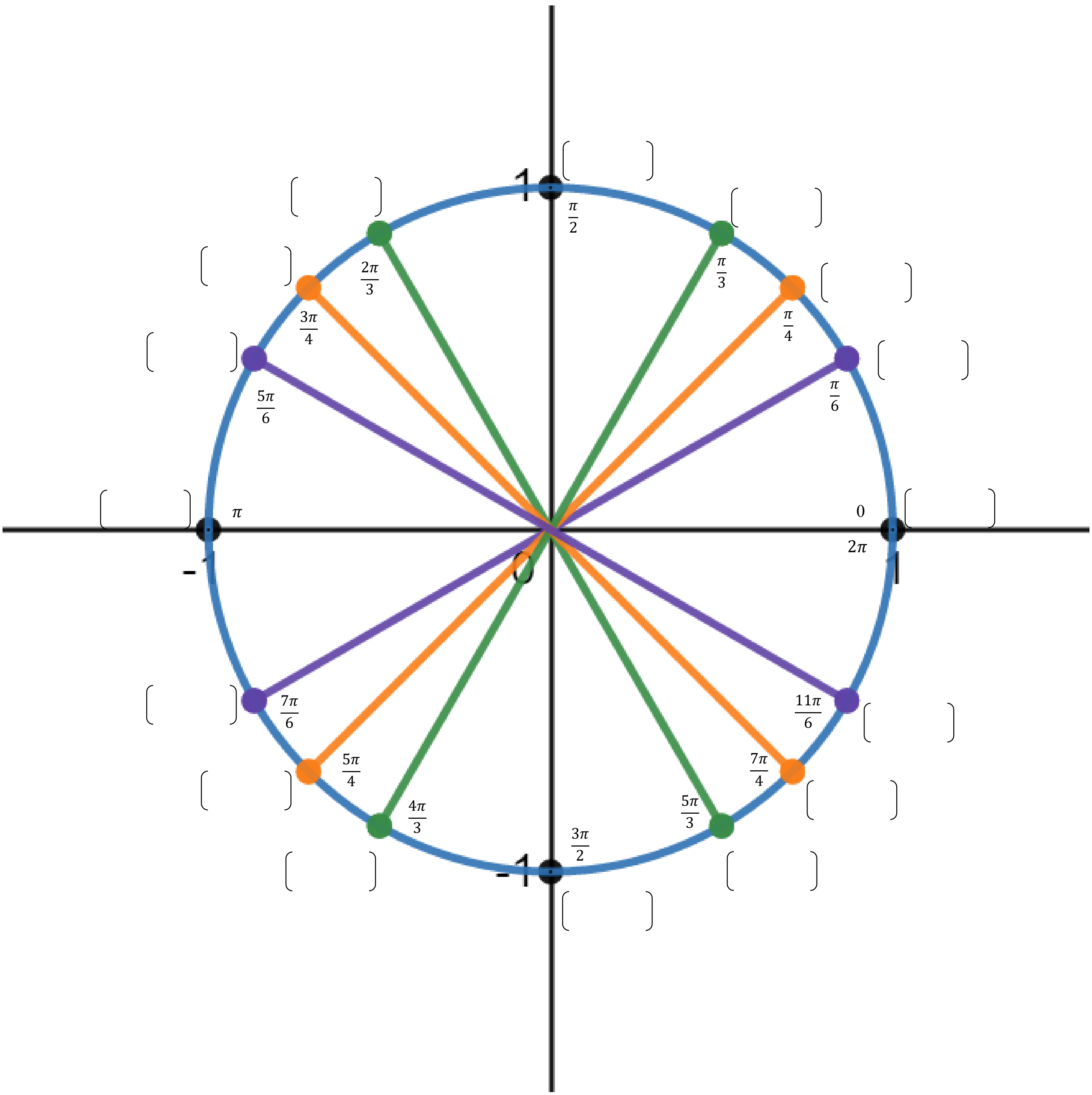
Desmos: The Unit Circle 2
https://www.desmos.com/calculator/p6cy3bxkkn

The Unit Circle and the Special Angles

Color code the reference angles:

- **Purple**: Reference angle 30°
- **Orange**: Reference angle 45°
- **Green**: Reference angle 60°

Exercise 2: Write the coordinates of the points on the unit circle in the brackets below.



Exercise 3: Find the value of the given trigonometric function without using a calculator.

a) $\tan 240°$

b) $\sin \frac{5\pi}{6}$

c) $\sec 135°$

d) $\cot \frac{2\pi}{3}$

e) $\csc 300°$

f) $\cos \frac{7\pi}{4}$

Exercise 4: Find the value of the trigonometric functions of other special angles.

a) $\csc 0°$

b) $\tan \pi$

c) $\sec 0°$

d) $\cot \frac{\pi}{2}$

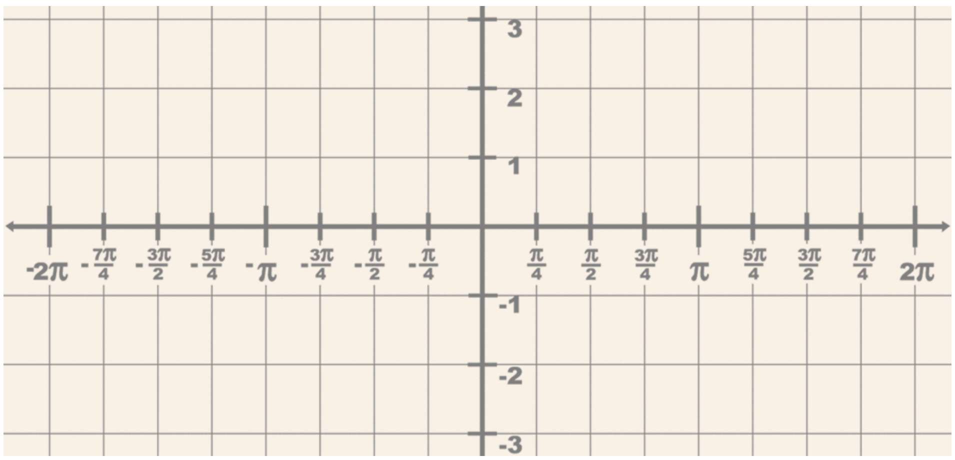
e) $\cot 180\,°$

f) $\sec \frac{3\pi}{2}$

Section 4.2 Graphs of Sine and Cosine Functions

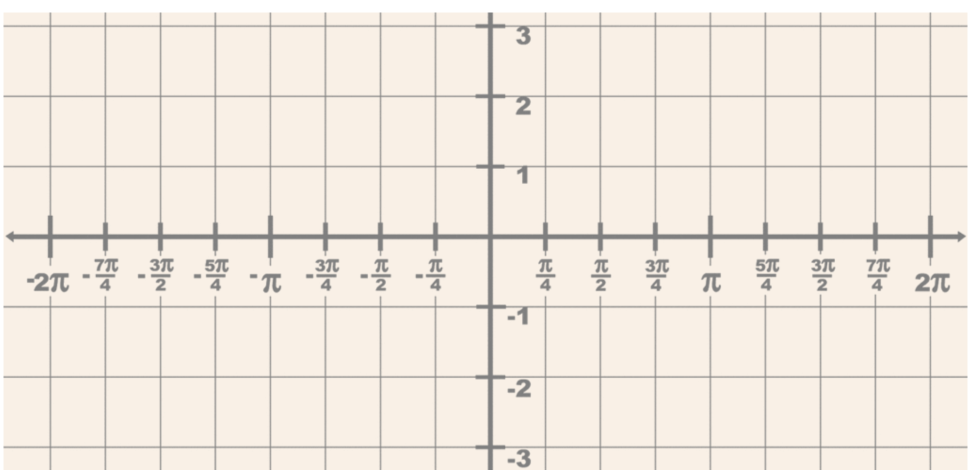
Fill out the table by determining the value of $y = \sin x$ for each given x. Plot the points on the graph.

x	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
$y = \sin x$								



Fill out the table by determining the value of $y = \cos x$ for each given x. Plot the points on the graph.

x	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
$y = \cos x$								



Characteristics of Sine and Cosine Graphs:

$$y = A\sin(Bx + C) + D \qquad\qquad y = A\cos(Bx + C) + D$$

Amplitude = A

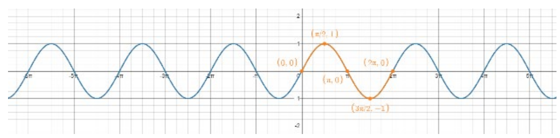
Period = $\frac{2\pi}{B}$

Horizontal Phase Shift = $-\frac{C}{B}$

Vertical Shift = D

Sine Graph

Desmos activity: Go to link https://www.desmos.com/calculator/aeouweh9xr



What happens to the sine graph when you change the amplitude $\boldsymbol{A}$?

____

What happens to the period of the sine graph when you change the value of $\boldsymbol{B}$?

____

What happens to the sine graph when you change the vertical shift $\boldsymbol{D}$?

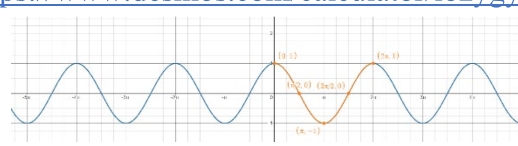
____

What happens to the sine graph when you change the value of $\boldsymbol{C}$?

____

Cosine Graph

Desmos activity: Go to link https://www.desmos.com/calculator/iozygyep5a



How is cosine graph similar to sine graph?

____

How is cosine graph different from sine graph?

____

Phase Shift

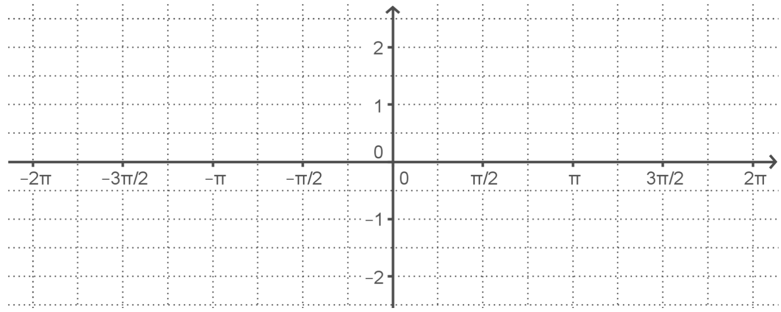
Desmos activity: Go to link https://www.desmos.com/calculator/rtn2hns4vm

How are sine and cosine graphs related by phase shift?

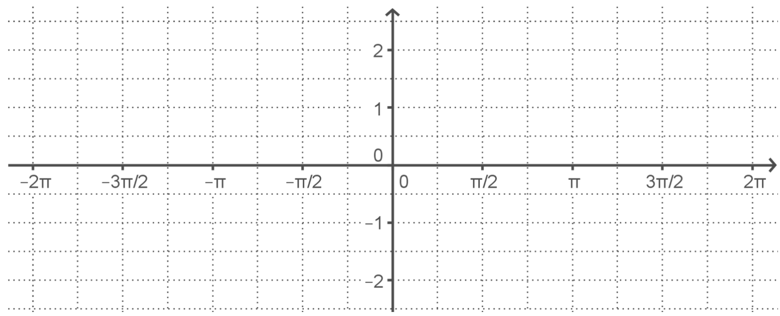
____

Sketch **one-period** of each of the following functions:

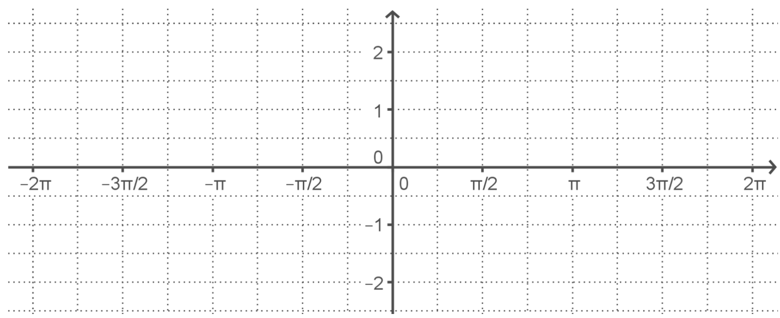
$y = 2\sin x$



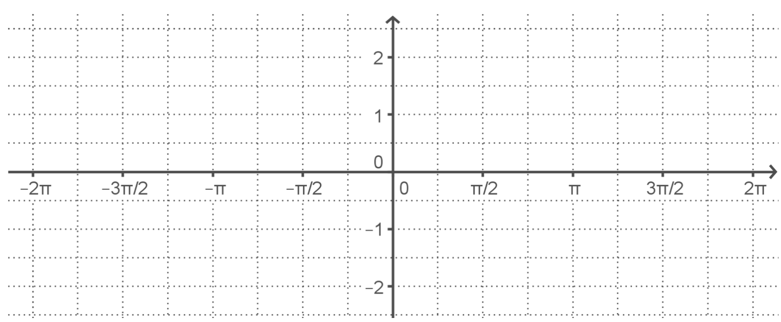
$y = \sin 2x$



$y = -\cos x$



$y = \cos\left(x + \frac{\pi}{2}\right)$



Section 4.3 Solving Trigonometric Equations

Example 1: Find all solutions in degrees in the interval $[0°,\ 360°)$ and in radians in the interval $[0,\ 2\pi)$. The solutions are all angles that satisfy the given trigonometric value.

a) Solve for x: $\sin x = \frac{\sqrt{3}}{2}$.

b) Solve for x: $\cos x = -\frac{\sqrt{2}}{2}$.

c) Solve for x: $\tan x = -1$.

The process of solving trigonometric equations is very similar to the process of solving algebraic equations.

Step 1: Solve for the trigonometric function by carrying out algebraic procedures such as adding, subtracting, multiplying, dividing, factoring, or the square root property

Step 2: Identify the angles that correspond to the values of the trigonometric function

Find all solutions of the given equation in degrees in the interval $[0°,\ 360°)$ and in radians in the interval $[0,\ 2\pi)$.

Example 1: Solve a trigonometric equation $2\sin x - 1 = 0$	Solve a corresponding algebraic equation $2x - 1 = 0$
Solve for $\sin x$: $2\sin x - 1 = 0$ $2\sin x = 1$ $\sin x = \dfrac{1}{2}$ Solutions are all angles that satisfy $\sin x = \frac{1}{2}$ Solutions in degrees: $x = 30°, 150°$ Solutions In radians: $x = \frac{\pi}{6}, \frac{5\pi}{6}$	**Solve for x:** $2x - 1 = 0$ $2x = 1$ $x = \dfrac{1}{2}$

Example 2: Solve a trigonometric equation $\tan^2 x - 1 = 0$	Solve a corresponding algebraic equation $x^2 - 1 = 0$
Solve for $\tan x$: $\tan^2 x - 1 = 0$ $(\tan x - 1)(\tan x + 1) = 0$ $\tan x - 1 = 0$ or $\tan x + 1 = 0$ $\tan x = 1$ or $\tan x = -1$ Solutions are all angles that satisfy $\tan x = 1$ or $\tan x = -1$ Find the solutions in degrees: Find the solutions In radians:	**Solve for x:** $x^2 - 1 = 0$ $(x-1)(x+1) = 0$ $x - 1 = 0$ or $x + 1 = 0$ $x = 1$ or $x = -1$

Exercise 1: Find all solutions of the following equations in degrees in the interval $[0°, 360°)$ and in radians in the interval $[0, 2\pi)$.

a) Solve: $2\cos x + 1 = 0$

b) Solve: $\sqrt{3}\tan x - 1 = 0$

c) Solve: $5\sin x - \sqrt{3} = 3\sin x$

d) Solve: $4\sin^2 x - 1 = 0$

e) Solve: $(2\cos x - \sqrt{3})(2\cos x - 1) = 0$

SECTION 4.1 SUPPLEMENTARY EXERCISES

Find the value of the given trigonometric function without using a calculator.

a) $\cos 150°$

b) $\sin 240°$

c) $\tan 225°$

d) $\csc 300°$

e) $\cot 120°$

f) $\sec 135°$

g) $\cos(-240°)$

h) $\tan(-150°)$

i) $\cot(-210°)$

j) $\csc(-60°)$

k) $\sec(-135°)$

l) $\sin 480°$

m) $\tan 405°$

n) $\csc 90°$

o) $\cot 180°$

p) $\sec(-90°)$

SECTION 4.3 SUPPLEMENTARY EXERCISES

Find all solutions of the following equations in degrees in the interval $[0°, 360°)$ and in radians in the interval $[0, 2\pi)$

a) Solve: $2\cos x - 1 = 0$

b) Solve: $2\sin x + 1 = 0$

c) Solve: $7\tan x - 7 = 0$

d) Solve: $\sqrt{3}\tan x - 1 = 0$

e) Solve: $5\cos x - \sqrt{3} = 3\cos x$

f) Solve: $2\sin x = 6\sin x - \sqrt{12}$

g) Solve: $4\cos^2 x - 1 = 0$

h) Solve: $5\tan^2 x - 5 = 0$

i) Solve: $3(\sin x + 2) = 3 - \sin x$

j) Solve: $(3\tan x + 1)(\tan x - 2) = 0$

k) Solve: $4(\cot x + 1) = 2(\cot x + 2)$

l) Solve: $3\cos^2 x - 4\cos x + 1 = 0$

m) Solve: $3\sin^2 x + 7\sin x + 2 = 0$

n) Solve: $2\cot^2 x - 13\cot x + 6 = 0$